VISUALIZATION OF NIL-GEOMETRY

Modelling Nil-geometry in Euclidean Space with Software Presentation

EMIL MOLNÁR — DÁVID PAPP

Budapest Universiti of Technology and Economics
Department of Geometry

Abstract. **Nil** geometry is a homogeneous 3-space derived from the Heisenberg matrix group, where the matrix multiplication provides the (non-commutative) addition of translations. A challenging problem is to visualize this geometry in the 3-diemsional Euclidean space. The task was to develop a software that simulates the effects of translations in **Nil** run-time and lets the user "play" with the model and demonstrate the most important properties of **Nil** geometry.

The very first problem is the visualization of the lattice points and lines in **Nil**. The integer coordinates of the lattice points are derived through the composition of the three basic translations (in the direction of the axes of the coordinate-system.) Connecting lattice points by straight lines yields a suitable visualization of the lattice — although "lines" in **Nil** geometry are not straight in the Euclidean model, in general.

The second important visualization problem is to draw the geodesics of **Nil** geometry. The created software enables the user to select the starting point and direction, then an approximating polygon of the chosen geodesic line is calculated and drawn on the screen.

The third problem is the visualization of the fundamental domain to the minimally presented discrete translation group of **Nil**. This is a topological tetrahedron with nonplane faces, according to the two generating translations. In this case, an approximating polyhedron is drawn.

The core of the program code is fully portable. The mathematical proecedures were written in C++, standard visualization operations (e.g. mapping model coordinates to screen coordinates) are performed using OpenGL routines. Only basic window management routines use Windows API.

1 The Nil space modelled in $\mathbf{E}^3 \subset \mathcal{P}^3$

In studying magnetic fields, WERNER HEISENBERG found his famous *real matrix group* $\mathbf{L}(\mathbf{R})$ whose left (row-column) multiplication by

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+x & c+xb+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

provided a new addition of points (translations)

$$(x, y, z) * (a, b, c) = (a + x, b + y, c + xb + z), \tag{2}$$

i.e. *the translations are not commutative*, in general.

The matrices $\mathbf{K}(z) \triangleleft \mathbf{L}$ of the form

$$\mathbf{K}(z) \ni \begin{pmatrix} 1 & 0 & z \\ & 1 & 0 \\ & & 1 \end{pmatrix} \mapsto (0,0,z), \tag{3}$$

however, constitute the *cyclic centre*, i.e. each of them is commuting with all elements of $\mathbf{L}$. The elements of $\mathbf{K}$ are called *fibre translations*, as well, and they can be visualized by straight lines, growing out from the points of the $(x,y,0)$ plane. Any *fibre line* is an orbit of a point $(x,y,0) \mapsto (x,y,z)$ under the fibre translations $\mathbf{K}(z)$, where $z \in \mathbf{R}$ being varied.

In the following we consider $\mathbf{L}$ as *projective collineation group* (see [LM92], but here) with *right actions* in homogeneous coordinates as follows

$$(1,a,b,c) \begin{pmatrix} 1 & x & y & z \\ & 1 & 0 & 0 \\ & & 1 & x \\ & & & 1 \end{pmatrix} = (1, x+a, y+b, z+bx+c). \tag{4}$$

The points of $\mathbf{Nil}$ will be visualized in $\mathbf{E}^3$ and embedded into the projective space $\mathcal{P}^3$, where the ideal points $(0,u,v,w)$, with direction vector (u,v,w), will be taken under the collineations in (4), as well.

Any plane $\boldsymbol{u} \sim (u_0,u_1,u_2,u_3)^T$, with linear equation for its points (row matrices) $\mathbf{x} \sim (x^0,x^1,x^2,x^3) \sim (1,x,y,z)$ ("$\sim$" means a freedom up to a non-zero $\mathbf{R}$ factor), i.e.

$$0 = \mathbf{x}\boldsymbol{u} = (x^0,x^1,x^2,x^3) \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} =$$

$$x^0u_0 + x^1u_1 + x^2u_2 + x^3u_3 \sim 1u_0 + xu_1 + yu_2 + zu_3, \tag{5}$$

is described by a linear form $\boldsymbol{u}$ (column matrix, upper T means transposition), again up to a non-zero $\mathbf{R}$ factor. The collineation in (4) for points induces the corresponding collineation for planes by *inverse matrix* (with *left action*) as follows

$$\begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -x & -y & xy-z \\ & 1 & 0 & 0 \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}. \tag{6}$$

Namely, this is the criterion, that any incident point and plane will be mapped under the collineation onto incident point and plane.

Now we can extend the translation group $\mathbf{L}$ defined by formulas (4) and (6) to a larger group $\mathbf{G}$ of collineations, preserving the fibering, that will be the (orientation preserving) *isometry group of* $\mathbf{Nil}$. We indicate how to introduce the rotation about the fibre over the origin about angle ω

by the usual matrix

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos\omega & \sin\omega & 0\\ 0 & -\sin\omega & \cos\omega & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{7}$$

leaving invariant the *infinitezimal arc-length-square*

$$(ds)^2 = (d\bar{x})^2 + (d\bar{y})^2 + (d\bar{z})^2 \tag{8}$$

as a positive definit quadratic differential form at the origin. By the *Lie theory* this will be extended to the rotation about the fibre over any point $(1, x, y, 0)$ by conjugacy (see (4) and (6))

$$\begin{pmatrix}1 & -x & -y & xy-z\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -x\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \cos\omega & \sin\omega & 0\\ 0 & -\sin\omega & \cos\omega & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & x & y & z\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & x\\ 0 & 0 & 0 & 1\end{pmatrix} = \tag{9}$$

$$\begin{pmatrix}1 & x(1-\cos\omega)+y\sin\omega & -x\sin\omega+y(1-\cos\omega) & -x^2\sin\omega+xy(1-\cos\omega)\\ 0 & \cos\omega & \sin\omega & x\sin\omega\\ 0 & -\sin\omega & \cos\omega & -x(1-\cos\omega)\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Moreover, we have the “pull-back transform”

$$(0, dx, dy, dz)\begin{pmatrix}1 & -x & -y & xy-z\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -x\\ 0 & 0 & 0 & 1\end{pmatrix} = (0, d\bar{x}, d\bar{y}, d\bar{z}) \tag{10}$$

for the basis differential forms at $(1, x, y, z)$ and at the origin, respectively. From this we obtain the *infinitezimal arc-length-square* by (8) at any point of **Nil** as follows

$$(dx)^2 + (dy)^2 + (-xdy + dz)^2 =$$

$$(dx)^2 + (1+x^2)(dy)^2 - 2x(dy)(dz) + (dz)^2 =: (ds)^2. \tag{11}$$

Hence we get the symmetric metric tensor field g on **Nil** by components, furthermore its inverse:

$$g_{ij} := \begin{pmatrix}1 & 0 & 0\\ 0 & 1+x^2 & -x\\ 0 & -x & 1\end{pmatrix}, \qquad g^{jk} := \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & x\\ 0 & x & 1+x^2\end{pmatrix}. \tag{12}$$

Thus **Nil** is a *homogeneous Riemann space* where the arc-length of any piecewise smooth curve can be computed by integration as usual for surface curves in the classical differential geometry.

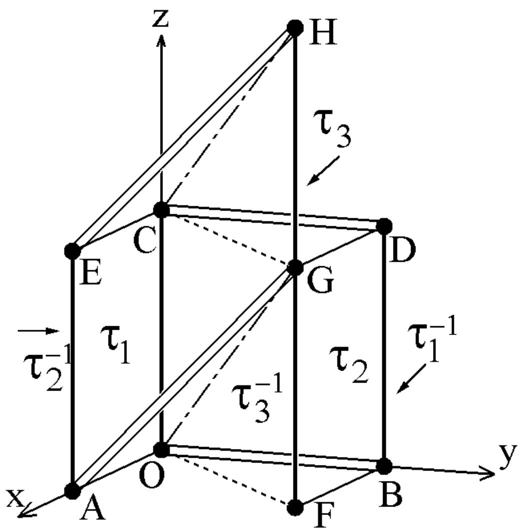

Figure 2.1: A fundamental domain $\tilde{\mathcal{F}}$ for $\mathbf{L}(\mathbf{Z})$, representing the $\mathbf{Nil}$ space form $\mathbf{Nil}/\mathbf{L}(\mathbf{Z})$.

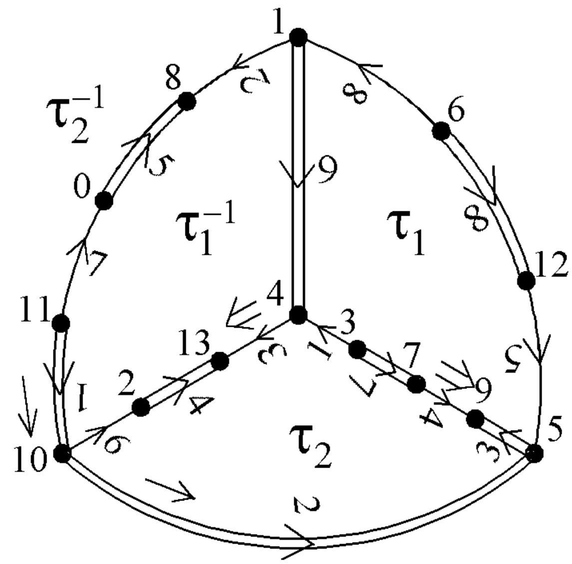

Figure 2.2: The minimally presenting fundamental tetrahedron $\tilde{\mathcal{T}}$ for $\mathbf{Nil}/\mathbf{L}(\mathbf{Z})$.

2 The discrete translation group $\mathbf{L}(\mathbf{Z})$

If we substitute *integers*, their set is denoted by $\mathbf{Z}$, into the formulas (1), (2) or (4) for x, y, z, then we get *discrete group actions* whose set will be denoted by $\mathbf{L}(\mathbf{Z})$, as integer lattice of $\mathbf{Nil}$.

As a surprising phenomenon, we illustrate the action of $\mathbf{L}(\mathbf{Z})$ on $\mathbf{Nil}$ in Figure 2.1 by a *fundamental domain* $\mathcal{F} = OABCDEFGH$. We remark that the Euclidean integer lattice may have a cube as fundamental domain, whose opposite side faces are mapped under the *three generating translations* [M92]. Now (4) provides us the face pairing generators as follows

$$\begin{array}{rrcll}
\boldsymbol{\tau}_1: & OBDC =: \tau_1^{-1} & \mapsto & \tau_1 := AGHE, & \text{i.e.} \\
& (1,0,b,c) & \mapsto & (1,1,b,c{+}b) & 0 \le b \le 1, \quad 0 \le c \le 1; \\
\boldsymbol{\tau}_2: & OAEC =: \tau_2^{-1} & \mapsto & \tau_2 := BFGD; & \\
\boldsymbol{\tau}_3: & OAGFB =: \tau_3^{-1} & \mapsto & \tau_3 := CEHGD. &
\end{array} \tag{13}$$

Here the bent faces τ_3^{-1} and τ_3 are remarkable. Of course, e.g. the inverse translation $\boldsymbol{\tau}_3^{-1}$: $\tau_3 \mapsto \tau_3^{-1}$ has also been defined.

These generators induce three $\mathbf{L}(\mathbf{Z})$ equivalence classes of edges, each class provides a so-called defining relation for the generators:

$$\begin{array}{lll}
= & \{OB, AG, EH, CD\} & \boldsymbol{\tau}_1\boldsymbol{\tau}_3\boldsymbol{\tau}_1^{-1}\boldsymbol{\tau}_3^{-1} = \mathbf{1} \quad (\textit{identity map}), \\
- & \{OA, BF, DG, CE\} & \boldsymbol{\tau}_2\boldsymbol{\tau}_3\boldsymbol{\tau}_2^{-1}\boldsymbol{\tau}_3^{-1} = \mathbf{1}, \\
\blacksquare & \{OC, AE, FG, GH, BD\} & \boldsymbol{\tau}_1\boldsymbol{\tau}_2\boldsymbol{\tau}_3\boldsymbol{\tau}_1^{-1}\boldsymbol{\tau}_2^{-1} = \mathbf{1},
\end{array} \tag{14}$$

as indicated in Figure 2.1. Now we only remark that any relation above can be read off a standard procedure (generalized Poincare algorithm, see [M92]): The image edge domains belonging to any edge class amount a complete tubular neighbourhood of each edge in the class.

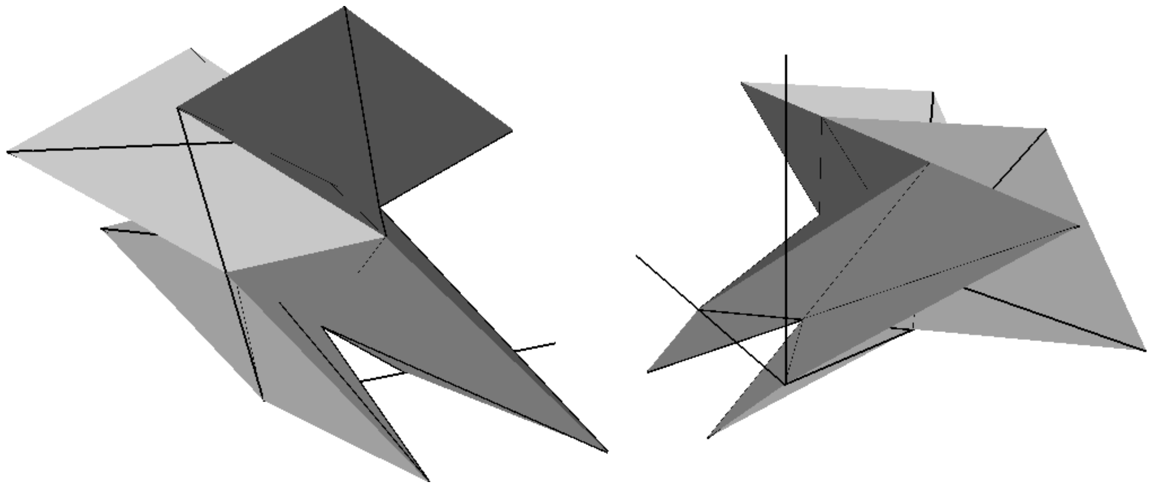

Figure 2.3: The minimally presenting domain for $\mathbf{Nil}$ in two positions (see Figure 2.2).

The vertices of $\mathcal{F}$ also fall into one equivalence class, and the image corner domains amount a ball-like neighbourhood of each vertex in the class. All these arguments imply that the fundamental domain $\tilde{\mathcal{F}}$, with face pairing identifications ($\tilde{}$), represents a compact $\mathbf{Nil}$ manifold or $\mathbf{Nil}$ space form, denoted by $\mathbf{Nil}/\mathbf{L}(\mathbf{Z})$.

The last relation of (14) provides $\boldsymbol{\tau}_3 = \boldsymbol{\tau}_2^{-1}\boldsymbol{\tau}_1^{-1}\boldsymbol{\tau}_2\boldsymbol{\tau}_1$ as a *commutator,* generating the centre $\mathbf{K}(\mathbf{Z})$ (as in (3)) of $\mathbf{L}(\mathbf{Z})$. Substituting $\boldsymbol{\tau}_3$ into the first two relations of (14), we get a *minimal presentation:*

$$\mathbf{L}(\mathbf{Z}) = \left\{\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 — \ \mathbf{1} = \boldsymbol{\tau}_2^2\boldsymbol{\tau}_1^{-1}\boldsymbol{\tau}_1^2\boldsymbol{\tau}_2^{-1}\boldsymbol{\tau}_1^{-1} = \boldsymbol{\tau}_1^{-1}\boldsymbol{\tau}_2\boldsymbol{\tau}_1\boldsymbol{\tau}_2^{-1}\boldsymbol{\tau}_1^{-1}\boldsymbol{\tau}_2^{-1}\boldsymbol{\tau}_1\boldsymbol{\tau}_2\right\} \qquad (15)$$

This minimal presentation has a geometrically realizing fundamental domain $\tilde{\mathcal{T}}$, a topological tetrahedron with face pairing generators $\boldsymbol{\tau}_1 : \tau_1^{-1} \mapsto \tau_1,\ \boldsymbol{\tau}_2 : \tau_2^{-1} \mapsto \tau_2$ as above (Figure 2.2).

This *Schlegel diagram* has a coordinate realization, analogously to Figure 2.1, with great freedom, but this is a *computer graphic problem* to solve. We have to produce the vertices of $\tilde{\mathcal{T}}$ with an appropriate starting vertex, first e.g. with the origin O, then its images as Figure 2.2 dictates:

$$\begin{gathered} O, \quad 1 := O^{\boldsymbol{\tau}_1}, \quad 2 := O^{\boldsymbol{\tau}_2}, \quad 3 := 1^{\boldsymbol{\tau}_1}, \quad 4 := 1^{\boldsymbol{\tau}_2}, \quad 5 := 2^{\boldsymbol{\tau}_1}, \quad 6 := 3^{\boldsymbol{\tau}_2^{-1}}, \quad 7 := 4^{\boldsymbol{\tau}_1}, \\ 8 := 4^{\boldsymbol{\tau}_1^{-1}}, \quad 9 := 5^{\boldsymbol{\tau}_2}, \quad 10 := 5^{\boldsymbol{\tau}_2^{-1}}, \quad 11 := 6^{\boldsymbol{\tau}_1^{-1}}, \quad 12 := 7^{\boldsymbol{\tau}_2^{-1}}, \quad 13 := 8^{\boldsymbol{\tau}_2}. \qquad (16) \end{gathered}$$

Then we form the edges. An appropriate centre, e.g. the *barycentre* of the above vertices of the face τ_1^{-1}, enables us to form the star-like face τ_1^{-1}, indeed. The $\boldsymbol{\tau}_1$ image of the former centre also provides the star-like face τ_1. Similarly, we can construct the faces τ_2^{-1} and τ_2 and the polyhedron $\tilde{\mathcal{T}}$ by computer. A simplicial subdivision of $\tilde{\mathcal{T}}$ can be produced by the barycentre of all vertices in (2) as a formal centre for $\tilde{\mathcal{T}}$.

This new polyhedron type shows how to apply our method in the group theory, and many new problems arise.

3 Nil geodesics

We are interested in determining the *geodesic curves* in our **Nil** geometry. As it is well-known, this curves are generally defined as *having locally minimal* (stationary) *arc length* between their any two (near enough) points.

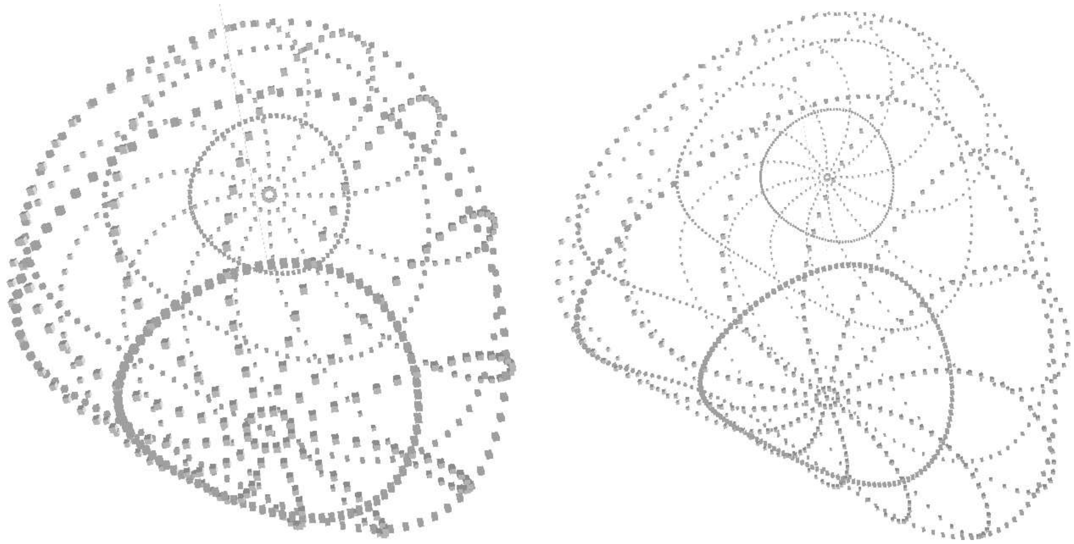

Figure 3.1: Geodesic balls in **Nil** with different radii.

Then it holds a second order differential equation (system)

$$\ddot{y}^k + \dot{y}^i \dot{y}^j \Gamma^k_{ij} = 0 \tag{17}$$

where $y^1(t) =: x(t)$, $y^2(t) =: y(t)$, $y^3(t) =: z(t)$ are the coordinate components of the parametrized geodesic curves, upper point means the derivation $\frac{d}{dt}$ by the parameter t, as usual. The Einstein-Schouten index conventions will be applied for recalling the general theory. Namely, the Levi-Civita connection by

$$\Gamma^k_{ij} = \frac{1}{2}\left(\frac{\partial g_{jl}}{\partial y^i} + \frac{\partial g_{li}}{\partial y^j} - \frac{\partial g_{ij}}{\partial y^l} \right) g^{lk} \tag{18}$$

can be expressed by (11) and (12) from the metric tensor field, by an easy but lengthy computation. Finally we obtain the system to solve

$$\begin{array}{lcll} (i) & \ddot{x} + \dot{y}\dot{y}(-x) + \dot{y}\dot{z} = 0 & \text{with} & x(0) = y(0) = z(0) = 0 \\ (ii) & \ddot{y} + \dot{x}\dot{y}(x) + \dot{x}\dot{z}(-1) = 0 & & \dot{x}(0) = c\cos\alpha \quad \dot{y}(0) = c\sin\alpha \\ (iii) & \ddot{z} + \dot{x}\dot{y}(x^2-1) + \dot{x}\dot{z}(-x) = 0 & & \dot{z}(0) = w \end{array} \tag{19}$$

as initial values. For simplicity we have chosen the origin as starting point, by the homogeneity of **Nil** this can be assumed, because of (1) we can transform a curve into an another starting point.

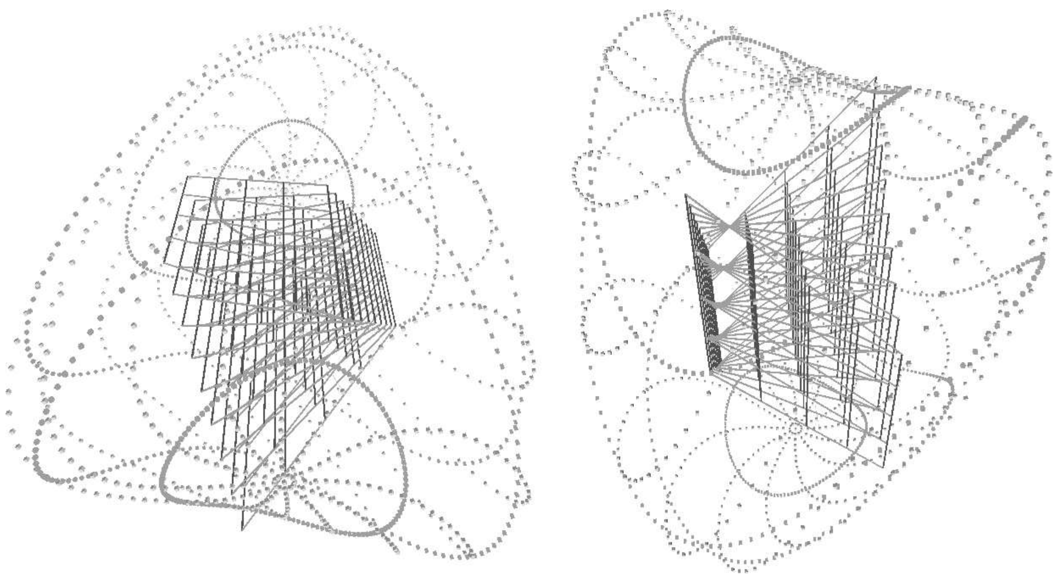

Figure 3.2: Geodesic balls in the **Nil** lattice.

Here we can introduce the arc length parameter

$$s = \sqrt{c^2 + w^2} \cdot t, \quad \text{moreover,} \quad w = \sin\vartheta, \quad c = \cos\vartheta, \quad -\frac{\pi}{2} \leq \vartheta \leq \frac{\pi}{2}, \tag{20}$$

i.e. unit velocity can be assumed.

We remark that there is no more simple relation among the distance s, and the coordinates (x, y, z), as it has been in the Euclidean space.

In the following form we obtain the solution

$$\begin{gathered} w \neq 0 \\ x(t) = \frac{2c}{w} \sin\frac{wt}{2} \cos\Big(\frac{wt}{2} + \alpha\Big), \qquad y(t) = \frac{2c}{w} \sin\frac{wt}{2} \sin\Big(\frac{wt}{2} + \alpha\Big), \\ z(t) = wt \cdot \Big\{1 + \frac{c^2}{2w^2}\Big[\Big(1 - \frac{\sin(2wt + 2\alpha) - \sin 2\alpha}{2wt}\Big) + \\ + \Big(1 - \frac{\sin(wt)}{wt}\Big) - \Big(1 - \frac{\sin(wt + 2\alpha) - \sin 2\alpha}{2wt}\Big)\Big]\Big\} = \\ = wt \cdot \Big\{1 + \frac{c^2}{2w^2}\Big[\Big(1 - \frac{\sin(wt)}{wt}\Big) + \frac{1 - \cos(wt)}{wt} \cdot \sin(wt + 2\alpha)\Big]\Big\} \end{gathered} \tag{21}$$

as a helix-like geodesic curve.

$$\begin{gathered} c = 0 \quad \text{leads to} \quad (x, y, z) = (0, 0, wt) \quad \text{as solution;} \\ w = 0 \quad \text{leads to} \quad x = c \cdot \cos\alpha \cdot t, \quad y = c \cdot \sin\alpha \cdot t, \\ z = \frac{1}{2} c^2 \cos\alpha \sin\alpha \cdot t^2 \end{gathered} \tag{22}$$

as a parabola on the hyperbolic paraboloid surface

$$2 \cdot Z - XY = 0. \tag{23}$$

Again, a nice computer visualization problem arises: *Determine the sphere of radius r in the* **Nil** *geometry!* The Euclidean analogies can help!?

We thank our colleagues ATTILA BÖLCSKEI and ISTVÁN PROK for preparing the manuscript and for designing the figures.

References

[LM92] P. LEDNECZKI – E. MOLNÁR: Projective geometry in engineering. *Periodica Polytechnica Ser. Mechanical Engineering* **39,** No. 1 (1995), 43–60.

[M92] E. MOLNÁR: Some old and new aspects on the crystallographic groups. *Periodica Polytechnica Ser. Mechanical Engineering* **36,** Nos. 3-4 (1992), 191–218.

[M97] E. MOLNÁR: The projective interpretation of the eight 3-dimensional homogeneous geometries. *Beiträge zur Algebra und Geometrie (Contributions to Algebra and Geometry)* Vol. **38** (1997), No. 2, 261–288.

[M03] E. MOLNÁR: On **Nil** geometry. *Periodica Polytechnica Ser. Mechanical Engineering* **47,** No. 1 (to appear).