Bilinear Optimality Constraints for the Cone of Positive Polynomials and Related Cones

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Abstract

For a proper cone $\mathcal{K} \subset \mathbb{R}^n$ and its dual cone $\mathcal{K}^*$ the complementary slackness condition $\mathbf{x}^T\mathbf{s} = 0$ defines an n-dimensional manifold $C(\mathcal{K})$ in the space $\{\ (\mathbf{x}, \mathbf{s}) \mid \mathbf{x} \in \mathcal{K},\ \mathbf{s} \in \mathcal{K}^*\ \}$. When $\mathcal{K}$ is a symmetric cone, this fact translates to a set of n bilinear optimality conditions satisfied by every $(\mathbf{x}, \mathbf{s}) \in C(\mathcal{K})$. This proves to be very useful when optimizing over such cones, therefore it is natural to look for similar optimality conditions for non-symmetric cones. In this paper we examine several well-known cones, in particular the cone of positive polynomials $\mathcal{P}_{2n+1}$ and its dual, the closure of the moment cone $\mathcal{M}_{2n+1}$. We show that there are exactly four linearly independent bilinear identities which hold for all $(\mathbf{x}, \mathbf{s}) \in C(\mathcal{P}_{2n+1})$, regardless of the dimension of the cones. For nonnegative polynomials over an interval or half-line there are only two linearly independent bilinear identities. These results are extended to trigonometric and exponential polynomials.

Introduction

In this paper we examine the complementarity conditions for convex cones. In particular, we are interested in those cones where complementarity can be expressed using bilinear relations. Our main result is that the complementarity conditions for the cone of positive polynomials and its dual, the closure of the moment cone over the real line, *cannot* be represented by bilinear relations alone. A similar result holds for the cone of positive polynomials over a given closed interval.

The cone of positive polynomials is a non-symmetric cone with many practical applications such as shape-constrained regression and the approximation of nonnegative functions (see for example [2, 11]).

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It is well-known that positive polynomials over the real line are precisely those polynomials that can be written as the sum of squares of other polynomials. This property directly leads to the expression of the cone of positive polynomials as a linear image or preimage of the cone of positive semidefinite matrices, see for example [10]. For instance, optimization over the cone of positive polynomials of degree $2n$ can be expressed as the dual of a semidefinite program over $n \times n$ Hankel matrices [3]. However, this approach may significantly increase the size of the problem and introduce degeneracy. This motivates us to look for solution methods and optimality conditions which directly apply to the cone of positive polynomials.

As a first step we wish to find as simple complementary slackness conditions as is possible for the positive polynomials and the moment cones. For instance, in linear programming complementary slackness conditions are given by $x_i s_i = 0$ where x_i are the primal variables and s_i are the dual slack variables. In semidefinite programming (SDP) the complementary slackness theorem is given by $XS + SX = 0$, where, again, X is the primal matrix variable and S is the dual slack matrix. Finally for second order cone programming (SOCP) we have $\langle \mathbf{x}, \mathbf{s} \rangle = 0$ and $s_0 x_i + s_i x_0 = 0$ (see the next section for more details). All of these relations are bilinear in the primal and dual slack variables. This property turns out to be essential in the design of primal-dual interior point algorithms. Furthermore, these bilinear forms make the machinery of certain algebraic structures available to help the understanding and improvement of such algorithms; this is especially true for SDP and SOCP.

According to a result of Güler, for every closed, pointed, convex cone $\mathcal{K}$ and its dual cone $\mathcal{K}^*$, the complementarity set $C(\mathcal{K})$, that is, the set of vector pairs $(\mathbf{x}, \mathbf{s}) \in \mathbb{R}^{2n}$, where $\mathbf{x} \in \mathcal{K}$, $\mathbf{s} \in \mathcal{K}^*$ and $\langle \mathbf{x}, \mathbf{s} \rangle = 0$, is an n-dimensional manifold. In many cases, this fact translates to a computationally tractable set of n equations $f_i(\mathbf{x}, \mathbf{s}) = 0$ $(i = 1, \ldots, n)$, which form the basis of complementary slackness theorems in optimization problems. Thus, it is an interesting endeavor to seek the simplest and most natural expressions for such relations. In fact, if it is at all possible to represent complementarity relations with bilinear forms, then that would be ideal, because potentially primal-dual interior point algorithms can be designed for such cones. Furthermore, bilinear relations induce algebras, and properties of these algebras may shed light on the properties of these cones and optimization problems over them [12].

In this paper we develop some techniques for proving that for certain cones, bilinear relations are not sufficient to express complementary slackness. In particular, we show this for positive polynomials, positive trigonometric polynomials, and positive exponential polynomials. The method we apply relies on results allowing the parametrization of the boundaries of these cones based on the theory of Chebyshev systems [7].

The paper is structured as follows: in Section 1 we present some fundamental concepts and results related to complementarity for proper cones, and introduce the notion of algebraic cones. In Section 2 we present a simple proof template for showing that cones are not algebraic. In the process we show a few simple cones that are not algebraic. We review necessary background information about the cone of positive polynomials $\mathcal{P}_{2n+1}$ and its dual, the closure of the moment cone $\mathcal{M}_{2n+1}$ in Section 3. Section 4 contains our main results concerning bilinear optimality constraints where we show that for the cone positive polynomials there are exactly four linearly independent bilinear complementarity relations. We also show that for the cone of positive polynomials over an interval there are exactly two such relations. Finally in Section 5 we use the notion of algebraic equivalence to show that several more cones of functions are not algebraic.

Notation

For a polynomial represented by the vector of its coefficients $\mathbf{p} = (p_0, \ldots, p_n)$ the corresponding polynomial function is denoted by $p(t) = p_0 + p_1 t + p_2 t^2 + \cdots + p_n t^n$. For a real $t \in \mathbb{R}$ and nonnegative integer n, $\mathbf{c}_{n+1}(t)$ denotes the moment vector $(1, t, \ldots, t^n)^\top$.

Throughout the paper we adopt the following convention: if for a range of indices the lower bound is greater than the upper bound, the range is considered to be empty.

The convex hull of a set $S \subset \mathbb{R}^n$ is denoted by $\operatorname{conv}(S)$, the closure of S is denoted by $\bar{S}$.

The linear space spanned by vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is denoted by $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$.

The inner product of vectors $\mathbf{x}$ and $\mathbf{s}$ is denoted by $\langle \mathbf{x}, \mathbf{s} \rangle = \mathbf{x}^T \mathbf{s}$.

The parity of an integer m is denoted by $m \pmod 2 = \begin{cases} 0 \text{ if } m \text{ is even} \\ 1 \text{ if } m \text{ is odd} \end{cases}$.

For a matrix $A = (a_{ij})_{m \times n}$, $\operatorname{vec}(A) \stackrel{\text{def}}{=} (a_{11}, \ldots, a_{n1}, a_{12}, \ldots, a_{n2}, \ldots, a_{mn})^\top$. For two column vectors $\mathbf{u}$ and $\mathbf{v}$, their Kronecker product is defined to be $\mathbf{u} \otimes \mathbf{v} \stackrel{\text{def}}{=} \operatorname{vec}(\mathbf{u}\mathbf{v}^\top)$.

1 Algebraic Cones

Let $\mathcal{K}$ be a proper cone in $\mathbb{R}^n$ (that is, a closed, pointed, and convex cone with nonempty interior in $\mathbb{R}^n$), and let

$$\mathcal{K}^* = \{\mathbf{z} \mid \langle \mathbf{x}, \mathbf{z} \rangle \geq 0, \quad \forall \, \mathbf{x} \in \mathcal{K}\}$$

be its *dual cone*. A pair of vectors $(\mathbf{x}, \mathbf{s})$, $\mathbf{x} \in \mathcal{K}$, $\mathbf{s} \in \mathcal{K}^*$ is said to satisfy the *complementary slackness conditions* with respect to $\mathcal{K}$ if $\langle \mathbf{x}, \mathbf{s} \rangle = 0$. We are interested in the following set:

Definition 1 *Let $\mathcal{K}$ be a proper cone, and $\mathcal{K}^*$ its dual. Then the set*

$$C(\mathcal{K}) = \{(\mathbf{x}, \mathbf{s}) \mid \mathbf{x} \in \mathcal{K}, \mathbf{s} \in \mathcal{K}^*, \langle \mathbf{x}, \mathbf{s} \rangle = 0\}$$

is called the complementarity set *of $\mathcal{K}$.*

Since for every proper cone $(\mathcal{K}^*)^* = \mathcal{K}$, it is immediate from the definition that $C(\mathcal{K})$ and $C(\mathcal{K}^*)$ are congruent: one can be obtained from the other by exchanging the first and last n coordinates.

The following theorem underlies the complementary slackness theorem for all convex optimization problems.

Theorem 2 *For each proper cone $\mathcal{K}$ in $\mathbb{R}^n$, $C(\mathcal{K})$ is an n-dimensional manifold homeomorphic to $\mathbb{R}^n$.*

A simple proof of this statement due to O. Güler [6] is given in Appendix.

To see the implications of this result for optimization problems over affine images or pre-images of proper cones, consider the following pair of dual *cone-LP* problems:

$$\begin{array}{llll} & \textbf{Primal} & & \textbf{Dual} \\ \inf & \langle \mathbf{c}, \mathbf{x} \rangle & \sup & \langle \mathbf{y}, \mathbf{b} \rangle \\ \text{s.t.} & A\mathbf{x} = \mathbf{b} & \text{s.t.} & A^\top \mathbf{y} + \mathbf{s} = \mathbf{c} \\ & \mathbf{x} \in \mathcal{K} & & \mathbf{s} \in \mathcal{K}^* \end{array} \tag{1}$$

It is easy to see that for any feasible solution $\mathbf{x}$ of the **Primal** problem and any feasible solution $(\mathbf{y}, \mathbf{s})$ of the **Dual** problem the quantities $\langle \mathbf{c}, \mathbf{x} \rangle - \langle \mathbf{y}, \mathbf{b} \rangle$ and $\langle \mathbf{x}, \mathbf{s} \rangle$ are equal and nonnegative. The *strong duality theorem* for cone-LP problems states the following: Under certain regularity conditions, if both the **Primal** and **Dual** problems are feasible, then inf and sup can be replaced by min and max. Moreover, the optimal objective values are equal, i.e., $\langle \mathbf{c}, \mathbf{x}^* \rangle - \langle \mathbf{y}^*, \mathbf{b} \rangle = \langle \mathbf{x}^*, \mathbf{s}^* \rangle = 0$. It follows that at the optimum we have $(\mathbf{x}^*, \mathbf{s}^*) \in C(\mathcal{K})$. Since $C(\mathcal{K}) \in \mathbb{R}^{2n}$ is n-dimensional, it is often possible to obtain a system of equations

$$\begin{array}{l} A\mathbf{x} = \mathbf{b} \\ A^\top \mathbf{y} + \mathbf{s} = \mathbf{c} \\ f_i(\mathbf{x}, \mathbf{s}) = 0 \quad \text{for } i = 1, \dots, n \end{array} \tag{2}$$

which is a square system, where $f_i(\mathbf{x}, \mathbf{s}) = 0$ are the complementarity equations. Many primal-dual algorithms for linear, second order and semidefinite programming problems, are based on strategies for solving this system of equations.

Let us examine some familiar examples.

Example 1 (Nonnegative orthant) When $\mathcal{K}$ is the nonnegative orthant, $\mathcal{K}^* = \mathcal{K}$. In this case if $\mathbf{x}$ and $\mathbf{s}$ contain only nonnegative components, and $\langle \mathbf{x}, \mathbf{s} \rangle = 0$, then we must have $x_i s_i = 0$ for $i = 1, \dots, n$. This is the basis of the familiar complementary slackness theorem in linear programming. ■

Example 2 (Positive semidefinite cone) If $\mathcal{K}$ is the cone of real, symmetric positive semidefinite matrices, then $\mathcal{K}^* = \mathcal{K}$. If both X and S are real symmetric positive semidefinite matrices, and $\langle X, S \rangle = \sum_{ij} X_{ij} S_{ij} = 0$, then it is easy to show that the matrix product $XS = 0$, or equivalently $XS + SX = 0$. This is the basis of the complementary slackness theorem in semidefinite programming. ■

Example 3 (Second order cones) Let $\mathcal{K} \in \mathbb{R}^{n+1}$ be the cone defined by all vectors $\mathbf{x}$ such that $x_0 \geq \|\overline{\mathbf{x}}\|$, where $\mathbf{x} = (x_0, x_1, \dots, x_n)$, $\overline{\mathbf{x}} = (x_1, \dots, x_n)$, and $\| \cdot \|$ is the Euclidean norm. This cone is also self-dual. Now if $\mathbf{x}, \mathbf{s} \in \mathcal{K}$ and $\langle \mathbf{x}, \mathbf{s} \rangle = 0$, then from Cauchy-Schwarz-Bunyakovsky inequality it follows that $x_0 s_i + x_i s_0 = 0$ for $i = 1, \dots, n$. These relations along with $\langle \mathbf{x}, \mathbf{s} \rangle = 0$ are the basis of the complementary slackness theorem for the *second order cone programming* problem. ■

Example 4 (L_p cones) Generalizing the previous example, suppose instead the cone $\mathcal{K}_p$ consists of vectors $\mathbf{x}$ such that $x_0 \geq \|\overline{\mathbf{x}}\|_p$, where $\|\cdot\|_p$ is the L_p norm for some real number $p > 1$. Then it is known that the dual cone is $\mathcal{K}_q$ where $\frac{1}{p} + \frac{1}{q} = 1$. In this case one can deduce from Hölder's inequality that if $\mathbf{x} \in \mathcal{K}_p$ and $\mathbf{s} \in \mathcal{K}_q$ and $\langle \mathbf{x}, \mathbf{s} \rangle = 0$, then $s_0^q \, |x_i|^p - x_0^p \, |s_i|^q = 0$ for $i = 1, \ldots, n$. ■

Example 5 (L_1 and L_∞ cones) A limiting case of the previous example is when $p = 1$ (and thus $q = \infty$). Here $\mathcal{K}_1$ consists of vectors $\mathbf{x}$ such that $x_0 \geq |x_1| + \cdots + |x_n|$, and $\mathcal{K}_\infty$ consists of vectors $\mathbf{s}$ where $s_0 \geq \max_i |s_i|$. In this case, if $\mathbf{x} \in \mathcal{K}_1$, $\mathbf{s} \in \mathcal{K}_\infty$ and $\langle \mathbf{x}, \mathbf{s} \rangle = 0$, then $x_i(s_0 - |s_i|) = 0$ for $i = 1, \ldots, n$. ■

Recall that an *algebra* is a linear space with an additional multiplication operation: $\mathbf{x} \cdot \mathbf{y} = \mathbf{z}$ defined on its vectors. The main requirement is that the components of $\mathbf{z}$ be expressed as bilinear functions of $\mathbf{x}$, and $\mathbf{y}$; in algebraic terms this multiplication must satisfy the distributive law; see for example [12]. Therefore, there are matrices Q_i such that $z_i = \mathbf{x}^\top Q_i \mathbf{y}$. If for a cone the complementarity relations can be exclusively expressed by bilinear forms, then, since these bilinear forms also define an algebra with multiplication, say "$\cdot$", the complementarity relations may be characterized by $\mathbf{x} \cdot \mathbf{s} = \mathbf{0}$. The machinery of this algebra may be useful in studying optimization problems over these cones. This motivates the following definitions.

Definition 3 *Let $\mathcal{K} \in \mathbb{R}^n$ be a proper cone. The $n \times n$ matrix Q is* a bilinear optimality condition for $\mathcal{K}$ *if every $(\mathbf{x}, \mathbf{s}) \in C(\mathcal{K})$ satisfies $\mathbf{x}^\top Q \mathbf{s} = 0$.*

Note that the set of all bilinear optimality conditions for $\mathcal{K}$, denoted by $\mathcal{Q}(\mathcal{K})$, is a linear subspace of $\mathbb{R}^{n \times n}$.

Definition 4 *A proper cone $\mathcal{K} \subseteq \mathbb{R}^n$ is called* algebraic *if there exist at least n linearly independent bilinear optimality conditions for $\mathcal{K}$.*

Remark 5 *An algebraic cone $\mathcal{K} \subseteq \mathbb{R}^n$ may have more than n bilinear optimality conditions, as the following example shows. Let $\mathcal{K}$ be the three-dimensional second order cone (see Example 3), and let*

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, Q_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Q_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, Q_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Then every $(\mathbf{x}, \mathbf{s}) \in C(\mathcal{K})$ satisfies $\mathbf{x}^\top Q_i \mathbf{s} = 0$, $i = 1, 2, 3, 4$. These four equations are linearly independent.

Since $C(\mathcal{K})$ and $C(\mathcal{K}^*)$ are congruent, the cone $\mathcal{K}^*$ is algebraic if and only if $\mathcal{K}$ is.

From the examples above we observe that the cones in Examples 1, 2, and 3 are algebraic. Note that in Example 5, even though $\mathcal{K}_1$ and $\mathcal{K}_\infty$ are polyhedral, the complementarity relations are not completely bilinear due to the absolute values. In Theorem 11 we show that $\mathcal{K}_1$ and $\mathcal{K}_\infty$ do not have any non-trivial bilinear complementarity relations.

The largest class of cones known to be algebraic are the *symmetric cones*. These are cones that are self-dual and homogeneous (that is, for any two points in the interior of the cone, there is a linear automorphism of the cone mapping the first point to the second one [4]). The cones in Examples 1, 2, and 3 are all symmetric. In addition, the cones of positive semidefinite complex Hermitian and quaternion Hermitian matrices are also symmetric. The second order cone, and the cones of positive semidefinite symmetric, complex Hermitian and quaternion Hermitian matrices, along with an exceptional 27 dimensional cone, are essentially the only symmetric cones; any other symmetric cones can be decomposed into direct sums of these five classes of cones.

Symmetric cones are intimately related to *Euclidean Jordan algebras*, see [4] and [8]. In such algebras the binary operation "$\circ$" is the abstraction of the operation $X \circ S = \frac{XS+SX}{2}$ in matrices. The properties of these algebras have played a major role in all aspects of optimization over such cones. In particular, design and analysis of interior point algorithms, duality, complementarity, and design of numerically efficient algorithms have been greatly simplified using the machinery of Jordan algebras. This is particularly true in the design of *primal-dual* interior point algorithms [5], [1].

There is an easy way to manufacture algebraic cones from other algebraic cones.

Definition 6 *The proper cones $\mathcal{K}$ and $\mathcal{L}$ are* algebraically equivalent *if there is a nonsingular (one-to-one and onto) linear transformation A such that $A\mathcal{K} = \mathcal{L}$.*

If two cones are algebraically equivalent, then one is algebraic if and only if the other one is. In fact, in the next section we introduce the concept of *bilinearity rank* of a cone and prove that this rank is invariant among all algebraically equivalent cones.

In the next two sections we develop techniques to prove certain cones are not algebraic.

2 A simple approach for proving cones are *not* algebraic

Recall that $\mathcal{Q}(K)$ denotes the linear space of all bilinear optimality conditions for $\mathcal{K}$, and consider the linear space

$$L(\mathcal{K}) \stackrel{\text{def}}{=} \operatorname{span}\{\mathbf{s}\mathbf{x}^\top \mid (\mathbf{x}, \mathbf{s}) \in C(\mathcal{K})\}.$$

Proposition 7 *For every proper cone $\mathcal{K}$ we have*

$$\dim(\mathcal{Q}(\mathcal{K})) = \text{co-dim}(L(\mathcal{K})).$$

Proof: Follows immediately from the identity $\mathbf{x}^\top Q\mathbf{s} = \langle \mathbf{s}\mathbf{x}^\top, Q^\top \rangle$. ■

Since by definition $X \in L(\mathcal{K})$ implies $\operatorname{trace} X = \langle X, I \rangle = 0$, the co-dimension of $L(\mathcal{K})$ as a subspace of $\mathbb{R}^{n\times n}$ is at least 1. Now if there are m linearly independent bilinear forms Q_i such that $\langle X, Q_i \rangle = 0$ for all $X \in L(\mathcal{K})$, then $\text{co-dim}(L(\mathcal{K})) \geq m$. Therefore, if we show $n^2 - k$ linearly independent matrices $X \in L(\mathcal{K})$, then this proves that there can be at most k bilinear forms in any characterization of $C(\mathcal{K})$. In particular, $\mathcal{K}$ is algebraic if and only if $\text{co-dim}(L(\mathcal{K})) \geq n$. Note that, as Remark 5 shows, it is possible that $\text{co-dim}(L(\mathcal{K})) > n$ for an algebraic cone $\mathcal{K}$.

Definition 8 *The quantity* $\dim(Q(\mathcal{K})) = \text{co-dim}(L(\mathcal{K}))$ *is called the* bilinearity rank *of* K *and is denoted by* $\beta(\mathcal{K})$*.*

The manifolds $C(\mathcal{K})$ and $C(\mathcal{K}^*)$ are congruent for every proper cone $\mathcal{K}$, implying $\beta(\mathcal{K}) = \beta(\mathcal{K}^*)$. Furthermore, we have:

Lemma 9 *If* $\mathcal{K}$ *and* $\mathcal{L}$ *are algebraically equivalent proper cones then* $\beta(\mathcal{K}) = \beta(\mathcal{L})$*.*

Proof: Let A be a nonsingular linear transformation such that $A\mathcal{K} = \mathcal{L}$. Then the dual cone of $A\mathcal{K}$ is the cone $A^{-\top}\mathcal{K}^*$. Furthermore, Q_i $(i = 1, \ldots, m)$ define linearly independent bilinear complementarity conditions for $\mathcal{K}$ if and only if $A^{-\top}Q_iA^\top$ $(i = 1, \ldots, m)$ define linearly independent bilinear complementarity conditions for $A\mathcal{K}$. ■

To derive our main results, we use the following simple fact.

Proposition 10 *If there are* k *pairs of vectors* $(\mathbf{x}_i, \mathbf{s}_i) \in C(\mathcal{K})$ *for* $i = 1, \ldots, k$*, such that the matrices* $\mathbf{s}_i\mathbf{x}_i^\top$ *are linearly independent, then* $\beta(\mathcal{K}) \geq k$*. In particular, if* $k > n^2 - n$*, then* $\mathcal{K}$ *is not algebraic.*

These results lead to the following template for proving certain cones are not algebraic: Suppose $\mathcal{K}$ is a proper cone in $\mathbb{R}^n$.

Step 1 Select a finite set S of orthogonal pairs of vectors $(\mathbf{x}, \mathbf{s})$, where $\mathbf{x}$ is a boundary vector of $\mathcal{K}$ and $\mathbf{s}$ is a boundary vector of $\mathcal{K}^*$.

Step 2 Form the matrix T whose rows are $\mathbf{x} \otimes \mathbf{s} = \text{vec}(\mathbf{s}\mathbf{x}^\top)$, $(\mathbf{x}, \mathbf{s}) \in S$.

Step 3 If $\operatorname{rank} T > n^2 - n$, then $\mathcal{K}$ is not algebraic. More generally, $\beta(\mathcal{K}) \leq n^2 - \operatorname{rank} T$.

To see how this template works let us show that the dual cones $\mathcal{K}_1, \mathcal{K}_\infty \subseteq \mathbb{R}^{n+1}$ from Example 5 are not algebraic for $n \geq 2$.

Theorem 11 $\beta(\mathcal{K}_1) = \beta(\mathcal{K}_\infty) = 1$*.*

Proof: As before, we assume that vectors are indexed from zero. We begin by introducing the following notation:

- $\mathbf{e}_i = (0, \ldots, 0, 1, 0 \ldots, 0)^\top \in \mathbb{R}^{n+1}$, with the single nonzero element in the ith position $(i = 0, \ldots, n)$,
- $\mathbf{f} = (1, \ldots, 1) \in \mathbb{R}^{n+1}$,
- $\mathbf{f}_i = (1, \ldots, 1, -1, 1, \ldots, 1) \in \mathbb{R}^{n+1}$, with all entries equal to 1 except in the ith position $(i = 0, \ldots, n)$,
- $\mathbf{f}_{ij} = (1, \ldots, 1, -1, 1, \ldots, 1, -1, , 1, \ldots 1) \in \mathbb{R}^{n+1}$ with all entries equal to 1 except in the ith and jth positions $(i, j = 0, \ldots, n)$.

The extreme rays of $\mathcal{K}_1$ are the $2n$ vectors $\mathbf{e}_0 \pm \mathbf{e}_i$ ($i = 1,\ldots,n$), while the extreme rays of $\mathcal{K}_\infty$ are the 2^n vectors of the form $(1, \pm 1, \pm 1, \ldots, \pm 1)^\top$. Specifically, for every $i, j = 1, \ldots, n$, the vectors $\mathbf{f}$, $\mathbf{f}_i$, and $\mathbf{f}_{ij}$ are among the extreme vectors of $\mathcal{K}_\infty$.

Let the set S (as described in Step 1 of the previous template) consist of the following orthogonal pairs $(\mathbf{x}, \mathbf{s})$ from $C(\mathcal{K}_1)$:

- $(\mathbf{e}_0 + \mathbf{e}_i, \mathbf{f}_i)$, $\quad i = 1, \ldots, n$,
- $(\mathbf{e}_0 - \mathbf{e}_i, \mathbf{f})$, $\quad i = 1, \ldots, n$,
- $(\mathbf{e}_0 + \mathbf{e}_i, \mathbf{f}_{ij})$, $\quad i, j = 1, \ldots, n, \; i \neq j$,
- $(\mathbf{e}_0 - \mathbf{e}_i, \mathbf{f}_j)$, $\quad i, j = 1, \ldots, n, \; i \neq j$,

and let the matrix T be constructed as in Step 2. The following vectors can be obtained as linear combinations of the rows of T.

$$\begin{aligned}
r_{0j} &= \mathbf{e}_0 \otimes \mathbf{e}_j = \frac{1}{4}\big((\mathbf{e}_0 + \mathbf{e}_1) \otimes \mathbf{f}_1 - (\mathbf{e}_0 + \mathbf{e}_1) \otimes \mathbf{f}_{1j} + (\mathbf{e}_0 - \mathbf{e}_1) \otimes \mathbf{f} - (\mathbf{e}_0 - \mathbf{e}_1) \otimes \mathbf{f}_j\big) \quad j = 1, \ldots, n, \\
r_{ij} &= \mathbf{e}_i \otimes \mathbf{e}_j = \frac{1}{4}\big((\mathbf{e}_0 + \mathbf{e}_i) \otimes \mathbf{f}_i - (\mathbf{e}_0 + \mathbf{e}_i) \otimes \mathbf{f}_{ij} - (\mathbf{e}_0 - \mathbf{e}_i) \otimes \mathbf{f} + (\mathbf{e}_0 - \mathbf{e}_i) \otimes \mathbf{f}_j\big) \quad i, j = 1, \ldots, n, \; i \neq j \\
r_{ii} &= -\mathbf{e}_0 \otimes \mathbf{e}_0 + \mathbf{e}_i \otimes \mathbf{e}_i = \sum_{\substack{1 \leq j \leq n \\ j \neq i}} r_{0j} - \frac{1}{2}\big((\mathbf{e}_0 + \mathbf{e}_i) \otimes \mathbf{f}_i + (\mathbf{e}_0 - \mathbf{e}_i) \otimes \mathbf{f}\big), \quad i = 1, \ldots, n \\
r_{i0} &= \mathbf{e}_i \otimes \mathbf{e}_0 = -(\mathbf{e}_0 + \mathbf{e}_i) \otimes \mathbf{f} + \sum_{j=1}^{n} r_{0j} - \sum_{\substack{1 \leq j \leq n \\ j \neq i}} r_{ij} - r_{ii}, \quad i = 1, \ldots, n.
\end{aligned}$$

Let $R \in \mathbb{R}^{[(n+1)^2 - 1] \times (n+1)^2}$ denote the matrix consisting of rows $\mathbf{r}_{01}, \mathbf{r}_{02}, \ldots, \mathbf{r}_{0n}, \mathbf{r}_{10}, \ldots, \mathbf{r}_{nn}$. Notice that by deleting the first column of R we obtain the identity matrix $I_{(n+1)^2 - 1}$. The rows of R were obtained as linear combinations of the rows of T, which in turn implies $\operatorname{rank} T \geq (n+1)^2 - 1$. In accordance with Step 3 of the previous template this completes the proof. ■

The template we used to prove Theorem 11 is a special case of the following, formally more general, framework:

Step 1 Select a set S of orthogonal pairs of vectors $(\mathbf{x}, \mathbf{s})$, where $\mathbf{x}$ is a boundary vector of $\mathcal{K}$ and $\mathbf{s}$ is a boundary vector of $\mathcal{K}^*$.

Step 2 Consider the set $\mathcal{T} = \{\mathbf{x} \otimes \mathbf{s} \,|\, (\mathbf{x}, \mathbf{s}) \in S\}$.

Step 3 If $\dim(\operatorname{span}(\mathcal{T})) > n^2 - n$, then $\mathcal{K}$ is not algebraic. More generally, $\beta(\mathcal{K}) \leq n^2 - \dim(\operatorname{span}(\mathcal{T}))$.

After presenting some necessary structural results in Section 3, we shall use these steps to prove our main results in Section 4.

3 Positive Polynomials and Moment Cones

Let us first introduce the cones of positive polynomials and moment cones:

Definition 12 *The* cone of positive polynomials *(also referred to as* cone of nonnegative polynomials*) of degree* $2n$

$$\mathcal{P}_{2n+1} \stackrel{\text{def}}{=} \left\{(p_0, \ldots, p_{2n}) \in \mathbb{R}^{2n+1} \mid p(t) = p_0 + p_1 t + p_2 t^2 + \cdots + p_{2n} t^{2n} \geq 0 \quad \forall t \in \mathbb{R}\right\}$$

consists of the coefficient vectors of nonnegative polynomials of degree $2n$*. Similarly, for real numbers* $a < b$*, the cone of* positive polynomials *(or* nonnegative polynomials*)* over the interval $[a, b]$ *of degree* n *is the cone*

$$\mathcal{P}^{[a,b]}_{n+1} \stackrel{\text{def}}{=} \left\{(p_0, \ldots, p_n) \in \mathbb{R}^{n+1} \mid p(t) = p_0 + p_1 t + p_2 t^2 + \cdots + p_n t^n \geq 0 \quad \forall t \in [a, b]\right\}.$$

The moment cone *of dimension* $2n + 1$ *is defined as*

$$\mathcal{M}_{2n+1} \stackrel{\text{def}}{=} \operatorname{conv}\left(\{\mathbf{c}_{2n+1}(t) \mid t \in \mathbb{R}\}\right), \text{ where } \mathbf{c}_{2n+1}(t) \stackrel{\text{def}}{=} (1, t, t^2, \ldots, t^{2n})^\top.$$

Similarly, the $(n+1)$*-dimensional moment cone over* $[a, b]$ *is defined as*

$$\mathcal{M}^{[a,b]}_{n+1} \stackrel{\text{def}}{=} \operatorname{conv}\left(\{\mathbf{c}_{n+1}(t) \mid t \in [a, b]\}\right).$$

Remark 13 *This is not the traditional definition of the moment cone. See [7] (Ch. VI) for the original definition and proof of its equivalence with the one given above.*

The cone of positive polynomials and the moment cone are closely related [7]:

Proposition 14 $\mathcal{P}^*_{2n+1} = \bar{\mathcal{M}}_{2n+1}$*. Similarly,* $(\mathcal{P}^{[a,b]}_{n+1})^* = \mathcal{M}^{[a,b]}_{n+1}$*.*

We will repeatedly use the following simple observation.

Proposition 15 *If* $\mathbf{p} \in \mathbb{R}^{n+1}$ *is the coefficient vector of a polynomial* p*, and* t *is real number, then* $p(t) = \langle \mathbf{p}, \mathbf{c}_{n+1}(t) \rangle$*. In particular,* $p(t) = 0$ *if and only if* $\langle \mathbf{p}, \mathbf{c}_{n+1}(t) \rangle = 0$*.*

In order to use the templates presented in Section 2 and prove that a cone $\mathcal{K}$ is not algebraic, it is useful to know the boundary or extreme rays of the cones $\mathcal{K}$ and $\mathcal{K}^*$. The extreme rays of $\mathcal{M}_{2n+1}$, and $\mathcal{M}^{[a,b]}_{n+1}$ are well known:

Proposition 16 ([7])

1. *The extreme vectors of* $\mathcal{M}_{2n+1}$ *are the vectors* $\alpha \mathbf{c}(t)$ *for every* $\alpha > 0$ *and* $t \in \mathbb{R}$*, and the vectors* $(0, \ldots, 0, \alpha)^\top$ *for every* $\alpha \geq 0$*.*
2. *The extreme rays of* $\mathcal{M}^{[a,b]}_{n+1}$ *are the vectors* $\alpha \mathbf{c}(t)$ *for every* $\alpha \geq 0$ *and* $t \in [a, b]$*.*

Finally, in the subsequent sections we will also use the following observations:

Proposition 17 ([7])

1. *Every real root of a nonnegative polynomial in $\mathcal{P}_{2n+1}$ is a multiple root with even multiplicity.*
2. *For polynomials in $\mathcal{P}_{n+1}^{[a,b]}$ every real root in the open interval (a,b) is a multiple root with even multiplicity.*

4 Main Results

In this section we show our main results, namely that neither the cone of positive polynomials over the real line, nor the cone of positive polynomials over a closed interval are algebraic. Moreover, we give the exact bilinearity rank for these cones.

To prove our main results we need the following elementary fact from linear algebra.

Lemma 18 *Let k be a positive integer and let $\mathcal{B} = \{b_1, \ldots, b_k\}$ be a set of linearly independent vectors in a real vector space. For a set $\{m_1, \ldots, m_k\} \subset \operatorname{span}(\mathcal{B})$ consider the coordinates $\alpha_{i,j} \in \mathbb{R}$ $(i, j = 1, \ldots, k)$ uniquely defined by the representations $m_i = \sum_{j=1}^{k} \alpha_{i,j} b_j$. (We refer to this as the $\mathcal{B}$-*representation *of m_i.) If the conditions*

$$\begin{array}{lll} \alpha_{i,i} \neq 0 & \text{for all} & 1 \leq i \leq k, \\ \alpha_{i,j} = 0 & \text{for all} & 1 \leq i < j \leq k \end{array}$$

hold, then the set $\{m_1, \ldots m_k\}$ is also linearly independent.

Proof: The claim follows immediately from the observation that the matrix $(\alpha_{i,j})_{k\times k}$ is lower triangular with a nonzero diagonal, and hence non-singular. ■

We are going to use the following, formally more general version of the above lemma:

Corollary 19 *Let $\mathcal{B} \subset \mathbb{R}[x_1, \ldots, x_n]$ be a finite set of linearly independent polynomials and consider a set $\mathcal{M} \subset \operatorname{span}(\mathcal{B})$ with coordinates $\alpha_{m,b}$ $(m \in \mathcal{M},\ b \in \mathcal{B})$ defined by the representations $m = \sum_{b\in\mathcal{B}} \alpha_{m,b} b$. Assume that there exists an injection $\varphi : \mathcal{B} \to \mathcal{M}$ and a linear order $\prec$ on $\varphi(\mathcal{B})$ such that*

$$\begin{array}{lll} \alpha_{\varphi(b),b} \neq 0 & \text{for all} & b \in \mathcal{B}, \\ \alpha_{\varphi(b),d} = 0 & \text{for all} & b, d \in \mathcal{B} \text{ satisfying } \varphi(b) \prec \varphi(d). \end{array}$$

Then $\dim(\operatorname{span}(\mathcal{M}(\mathbb{R}^n))) = |\mathcal{B}|$, *where* $\mathcal{M}(\mathbb{R}^n) \stackrel{\text{def}}{=} \{(m(\mathbf{x}))_{m\in\mathcal{M}} \,|\, \mathbf{x} \in \mathbb{R}^n\}$.

Proof: Let $k = |\mathcal{B}|$. It is well known that for a vector $P = (p_1, \ldots, p_k) \in (\mathbb{R}[x_1, \ldots, x_n])^k$ consisting of linearly independent polynomials we have $\dim(\operatorname{span}(P(\mathbb{R}^n))) = k$, therefore it suffices to find a k-element linearly independent subset of $\mathcal{M}$. As φ is injective, there exists an indexing $\mathcal{B} = \{b_1, \ldots, b_k\}$ such that $\varphi(b_1) \prec \cdots \prec \varphi(b_k)$. Let $m_i = \varphi(b_i) \in \mathcal{M}$ (for all $i = 1, \ldots, k$). It is easy to verify that the sets $\{b_1, \ldots, b_k\}$ and $\{m_1, \ldots, m_k\}$ satisfy the conditions of Lemma 18. Consequently the set $\{m_1, \ldots, m_k\} \subset \mathcal{M}$ is linearly independent, which implies our claim. ■

4.1 Positive polynomials over the real line

Theorem 20 *The cone $\mathcal{P}_{2n+1}$ is not algebraic, unless $n = 1$. More specifically, for every n, $\beta(\mathcal{P}_{2n+1}) \leq 4$.*

The second claim immediately implies the first. Note that when $n = 1$, we do have an algebraic cone algebraically equivalent to the cone of 2×2 positive semidefinite matrices.

Proof: Consider the matrix valued functions $M\colon \mathbb{R}^n \mapsto \mathbb{R}^{(2n+1)\times(2n+1)}$ defined as

$$M(t_1, \ldots, t_n) = \mathbf{c}\mathbf{p}^\top,$$

where $\mathbf{p} \in \mathcal{P}_{2n+1}$ is the coefficient vector of the polynomial $p(x) = \prod_{k=1}^n (x - t_k)^2$, and $\mathbf{c} = \mathbf{c}_{2n+1}(t_1) = (1, t_1, \ldots, t_1^{2n})$ is the moment vector corresponding to the first root of $\mathbf{p}$. It is easy to verify that the entries of $M = (m_{i,j})_{i,j=0}^{2n}$ satisfy the polynomial equation

$$\sum_{j=0}^{2n} m_{i,j} x^j \equiv t_1^i \prod_{k=1}^{n} (x - t_k)^2. \tag{3}$$

The polynomial $p(x)$ is clearly nonnegative everywhere, and $\mathbf{c}$ is a moment vector, furthermore, by Proposition 15, $\langle \mathbf{p}, \mathbf{c} \rangle = 0$. Therefore, following the general template of Section 2 (with $\mathbf{p}$ and $\mathbf{c}$ playing the roles of $\mathbf{x}$ and $\mathbf{s}$, and $M(\mathbb{R}^n)$ playing the role of $\mathcal{T}$), the theorem follows if $\dim(\operatorname{span}(M(\mathbb{R}^n))) = (2n+1)^2 - 4$. We show this equality using the sufficient condition presented in Corollary 19, with the set $\{m_{i,j}\}$ playing the role of set $\mathcal{M}$.

Let us define the n-variate polynomials $\Pi(k, \ell)$ by

$$\Pi(k,\ell)(t_1,\ldots,t_n) \stackrel{\text{def}}{=} \sum_{\substack{0 \leq \alpha_2,\ldots,\alpha_n \leq 2 \\ \alpha_2+\cdots+\alpha_n = \ell}} t_1^k \prod_{j=2}^{n} 2^{(\alpha_j \bmod 2)} t_j^{\alpha_j}, \tag{4}$$

whenever $0 \leq k \leq 2n+2$ and $0 \leq \ell \leq 2n-2$; for values of k and ℓ outside these ranges let us define $\Pi(k, \ell)$ to be the zero polynomial. Let $\mathcal{B}$ denote the set $\{\Pi(k, \ell) \mid 0 \leq k \leq 2n+2, 0 \leq \ell \leq 2n-2\}$. It follows from the definition that $|\mathcal{B}| = (2n+1)^2 - 4$, and that $\mathcal{B}$ is linearly independent, because no two polynomials share a common monomial. It remains to show that $\mathcal{M}$ is indeed a subset of $\operatorname{span}(\mathcal{B})$, and exhibit the injection φ and the linear order $\prec$ of Corollary 19.

The coefficient of $x^{2n-k-\ell}$ in the polynomial $\prod_{j=1}^n (x - t_j)^2$ is $\sum_{k=0}^{2} \Pi(k, \ell)$. From this observation it follows immediately that $\operatorname{span}(\mathcal{B})$ contains the entries of M; more specifically, for every $0 \leq i, j \leq 2n$,

$$m_{i,j} = \Pi(i, 2n - j) + \Pi(i + 1, 2n - 1 - j) + \Pi(i + 2, 2n - 2 - j). \tag{5}$$

We now introduce an injection $\varphi\colon \mathcal{B} \mapsto \mathcal{M}$ by defining its inverse (where it exists): let $m_{i,j}$ be the image of the polynomial

$$\varphi^{-1}(m_{i,j}) = q_{i,j} \stackrel{\text{def}}{=} \begin{cases} \Pi(i, 2n - j) & j \geq \max\{2, i\} \\ \Pi(i + 2, 2n - 2 - j) & j \leq \min\{i - 1, 2n - 2\} \\ \text{not defined} & \text{otherwise} \end{cases}. \tag{6}$$

In particular, we assign a polynomial to each entry $m_{i,j}$ of $\mathcal{M}$ except for $m_{0,0}$, $m_{0,1}$, $m_{1,1}$, and $m_{2n,2n-1}$, and we assign different polynomials to different entries of M, because if $q_{i_1,j_1} = q_{i_2,j_2}$ for some $(i_1, j_1) \neq (i_2, j_2)$ and $i_1 \le j_1$, then $j_1 \ge i_1$, $i_2 - 1 \ge j_2$, $i_1 = i_2 + 2$, and $2n - j_1 = 2n - 2 - j_2$, a contradiction, as the sum of these inequalities reduces to $-1 \ge 0$. Consequently, each $\Pi(k, \ell)$ is equal to $q_{i,j}$ for precisely one pair (i, j), therefore φ is indeed an injection.

Equation (5) shows that the coefficient of $q_{i,j}$ in the $\mathcal{B}$-representation of $m_{i,j}$ is 1, so using the notation of Corollary 19, $\alpha_{\varphi(\Pi(k,\ell)),\Pi(k,\ell)} = 1$ for all $\Pi(k, \ell) \in \mathcal{B}$.

Let us define a linear order $\succ$ on $\varphi(\mathcal{B})$ in the following way: $m_{i_1,j_1} \succ m_{i_2,j_2}$ precisely when one of the following three conditions holds:

1. $i_1 - j_1 \ge 1 > i_2 - j_2$;
2. $i_1 - j_1 \ge 1$, $i_2 - j_2 \ge 1$, and either $i_1 > i_2$, or $i_1 = i_2$ but $j_1 < j_2$;
3. $i_1 - j_1 < 1$, $i_2 - j_2 < 1$, and either $j_1 < j_2$, or $j_1 = j_2$ but $i_1 > i_2$.

An easy case analysis using Equations (5) and (6) shows that if $m_{i_1,j_1} \succ m_{i_2,j_2}$, then the coefficient of q_{i_1,j_1} in the $\mathcal{B}$-representation of m_{i_2,j_2} is zero:

1. If $i_1 - j_1 \ge 1 > i_2 - j_2$, then Equations (5) and (6) show that the three terms of m_{i_1,j_1} have higher degree than those of m_{i_2,j_2}, so in particular $\Pi(i_1 + 2, 2n - 2 - j_1)$ does not appear in the $\mathcal{B}$-representation of m_{i_2,j_2}.
2. If both $i_1 - j_1, i_2 - j_2 \ge 1$, then $i_1 + 2 > i_2 + 2$ or $2n - 2 - j_1 > 2n - 2 - j_2$, and by Equation (5) $\Pi(i_1 + 2, 2n - 2 - j_1)$ does not appear in the $\mathcal{B}$-representation of m_{i_2,j_2}.
3. If both $i_1 - j_1, i_2 - j_2 \le 0$, then $i_1 > i_2$ or $2n - j_1 > 2n - j_2$, and by Equation (5), $\Pi(i_1, 2n - j_1)$ does not appear in the $\mathcal{B}$-representation of m_{i_2,j_2}.

The injection $m_{i,j} \mapsto q_{i,j}$ and the linear order $\succ$ satisfy the conditions of Corollary 19, therefore, by Equation (6),

$$\dim(\mathrm{span}(M(\mathbb{R}^n))) = |\mathcal{B}| = (2n + 1)^2 - 4,$$

which completes the proof. ∎

4.2 Polynomials over a closed interval

We prove our theorem separately for polynomials of even and odd degree, since the different representations of the extreme rays would make a unified proof difficult. The main idea of the proofs is the same as in the proof of Theorem 20, however, the sets $\mathcal{M}$ and $\mathcal{B}$ are different, and the linear order $\prec$ also needs a more complicated definition.

In some cases it will be useful to restrict ourselves to the case when $[a, b] = [0, 1]$. This is without loss of generality: the same number of linearly independent bilinear optimality conditions exist for $\mathcal{P}^{[a,b]}_{n+1}$ as for $\mathcal{P}^{[0,1]}_{n+1}$, as the following proposition shows.

Proposition 21 *For every positive integer n and $a < b$, the cone $\mathcal{P}_{n+1}^{[a,b]}$ is algebraically equivalent to $\mathcal{P}_{n+1}^{[0,1]}$.*

Proof: The polynomial $p(t)$ is nonnegative over $[a, b]$ if and only if $q(t) = p\big(\frac{t-a}{b-a}\big)$ is nonnegative over $[0, 1]$. Furthermore, the coefficients of $q(t)$ can be obtained by a nonsingular linear transformation from of coefficients of p. ■

4.2.1 Polynomials of even degree

Theorem 22 *The cone $\mathcal{P}_{2n+1}^{[a,b]}$ is not algebraic. More specifically, for every n, $\beta(\mathcal{P}_{2n+1}^{[a,b]}) \leq 2$.*

Proof: Consider the matrix valued functions $M\colon \mathbb{R}^{n+2} \mapsto \mathbb{R}^{(2n+1)\times(2n+1)}$ defined as

$$M(t_1, \dots, t_n; \alpha, \beta) = \mathbf{c}_{2n+1}(t_1)\mathbf{p}^\top + \alpha\mathbf{c}_{2n+1}(a)\mathbf{p}_a^\top + \beta\mathbf{c}_{2n+1}(b)\mathbf{p}_b^\top,$$

where $\mathbf{p}, \mathbf{p}_a, \mathbf{p}_b \in \mathcal{P}_{2n+1}^{[a,b]}$ are the coefficient vectors of the polynomials $p(x) = \prod_{k=1}^n (x - t_k)^2$, $p_a(x) = x - a$, and $p_b(x) = b - x$, respectively. It is easy to verify that the entries of $M = (m_{i,j})_{i,j=0}^{2n}$ satisfy the polynomial equation

$$\sum_{j=0}^{2n} m_{i,j}x^j \equiv t_1^i \prod_{k=1}^{n} (x - t_k)^2 + \alpha a^i(x - a) + \beta b^i(b - x). \tag{7}$$

The polynomials $p(x)$, $p_a(x)$ and $p_b(x)$ are clearly nonnegative over $[a, b]$, and by Proposition 15, $\langle \mathbf{p}_a, \mathbf{c}_{2n+1}(a)\rangle = \langle \mathbf{p}_b, \mathbf{c}_{2n+1}(b)\rangle = \langle \mathbf{p}, \mathbf{c}_{2n+1}(t_1)\rangle = 0$. Consequently, following the general template of Section 2 (with $\mathbf{p}$, $\mathbf{p}_a$ and $\mathbf{p}_b$ playing the role of $\mathbf{x}$, and $\mathbf{c}_{2n+1}(t)$ playing the role of $\mathbf{s}$, and $M(\mathbb{R}^{n+2})$ playing the role of $\mathcal{T}$), the theorem follows if $\dim(\operatorname{span}(M(\mathbb{R}^{n+2}))) = (2n + 1)^2 - 2$. Finally, we will show this equality using the sufficient condition presented in Corollary 19, with the set $\{m_{i,j}\}$ playing the role of set $\mathcal{M}$.

For the rest of the proof, let us assume that $a = 0$, $b = 1$; Proposition 21 guarantees that this is without loss of generality.

With a slight abuse of notation, let us define the polynomials $\Pi(k, \ell)$ as in the proof of Theorem 20 (see Equation (4) and the subsequent paragraph), except that now every $\Pi(k, \ell)$ has formally two additional variables, α and β, even though they do not depend on these variables. Let $\mathcal{M}$ be the set of entries of the matrix M, and let $\mathcal{B}$ be the set

$$\mathcal{B} = \big\{\alpha, \beta\big\} \cup \big\{\Pi(k, \ell) \mid 0 \leq k \leq 2n + 2, 0 \leq \ell \leq 2n - 2\big\}.$$

The elements of $\mathcal{B}$ are considered as polynomials of $n + 2$ variables, $t_1, \dots, t_n, \alpha, \beta$. Again, it is immediate that the set $\mathcal{B}$ is linearly independent. It follows from these definitions that for every $0 \leq i, j \leq 2n$,

$$m_{i,j} = \Pi(i, 2n - j) + \Pi(i + 1, 2n - 1 - j) + \Pi(i + 2, 2n - 2 - j) + m'_{i,j}, \tag{8}$$

where

$$m'_{i,j} = \begin{cases} \beta & j = 0 \\ \alpha - \beta & i = 0, j = 1 \\ -\beta & i \geq 1, j = 1 \\ 0 & \text{otherwise} \end{cases}. \tag{9}$$

We now introduce an injection $\varphi \colon \mathcal{B} \mapsto \mathcal{M}$ by defining its inverse (where it exists): let $m_{i,j}$ be the image of the polynomial

$$\varphi^{-1}(m_{i,j}) = q_{i,j} \stackrel{\text{def}}{=} \begin{cases} \Pi(i, 2n - j) & j \geq \max\{2, i\} \\ \Pi(i + 2, 2n - 2 - j) & j \leq \min\{i - 1, 2n - 2\} \\ \beta & i = 0, j = 0 \\ \alpha & i = 0, j = 1 \\ \text{not defined} & \text{otherwise} \end{cases}. \tag{10}$$

In particular, we assign a polynomial to each entry except for $m_{1,1}$ and $m_{2n,2n-1}$, and we assign different polynomials to different entries of M, by an argument essentially identical to that in the proof of Theorem 20. Consequently, φ is indeed an injection, and Equations (8) and (9) show that the coefficient of $q_{i,j}$ in the $\mathcal{B}$-representation of $m_{i,j}$ is 1.

Let us define a linear order $\succ$ on $\varphi(\mathcal{B})$ in the following way: $m_{i_1,j_1} \succ m_{i_2,j_2}$ precisely when one of the following four conditions holds:

1. $(i_1, j_1) = (0, 1)$;
2. $i_1 - j_1 \geq 1 > i_2 - j_2$;
3. $i_1 - j_1 \geq 1$, $i_2 - j_2 \geq 1$, and either $i_1 > i_2$, or $i_1 = i_2$ but $j_1 < j_2$;
4. $i_1 - j_1 < 1$, $i_2 - j_2 < 1$, and either $j_1 < j_2$, or $j_1 = j_2$ but $i_1 > i_2$.

An easy case analysis using Equations (8), (9), and (10) shows that if $m_{i_1,j_1} \succ m_{i_2,j_2}$, then the coefficient of q_{i_1,j_1} in the $\mathcal{B}$-representation of m_{i_2,j_2} is zero.

This case analysis is essentially identical to the one in the proof of Theorem 20, except that now we also have to take care of $q_{0,0}$ and $q_{0,1}$. Hence we examine four cases in addition to the ones in the proof of Theorem 20:

1. If $(i_1, j_1) = (0, 1)$, then $q_{i_1,j_1} = \alpha$, and this polynomial has a nonzero coefficient exclusively in the $\mathcal{B}$-representation of $m_{0,1}$.
2. The case $(i_2, j_2) = (0, 1)$ is impossible.
3. If $(i_1, j_1) = (0, 0)$, then only the fourth condition is satisfied by (i_1, j_1), so $m_{i_1,j_1} \succ m_{i_2,j_2}$ implies $i_2 - j_2 < 1$, which in the light of (10) yields $j_2 \geq 2$. Consequently, by (9), $q_{0,0} = \beta$ has zero coefficient in the $\mathcal{B}$-representation of m_{i_2,j_2}.

4. If $(i_2, j_2) = (0,0)$, then $m_{i_1,j_1} \succ m_{i_2,j_2}$ implies $i_1 - j_1 \geq 1$, so the degree of q_{i_1,j_1} is larger than the degree of m_{i_2,j_2}. Consequently, q_{i_1,j_1} has zero coefficient in the $\mathcal{B}$-representation of $m_{0,0}$.

5. The cases in which both (i_1, j_1) and (i_2, j_2) are different from $(0,0)$ and $(0,1)$ are settled the same way as in the proof of Theorem 20.

The injection $m_{i,j} \mapsto q_{i,j}$ and the linear order $\succ$ satisfy the conditions of Corollary 19, therefore, by Equation (10),

$$\dim(\operatorname{span}(M(\mathbb{R}^{n+2}))) = |\mathcal{B}| = (2n+1)^2 - 2,$$

which completes the proof. ■

4.2.2 Polynomials of odd degree

We first prove our claim for the case $n = 1$.

Lemma 23 *The cone $\mathcal{P}_4^{[1,6]}$ is not algebraic. More specifically, $\beta(\mathcal{P}_4^{[1,6]}) \leq 2$.*

Proof: Following the first version of the template given in Section 2, we present a set S of 14 pairs of vectors $(\mathbf{x}, \mathbf{s}) \in C(\mathcal{P}_4^{[1,6]})$ such that the vectors $\operatorname{vec}(\mathbf{s}\mathbf{x}^\top)$ are linearly independent.

For every $i = 1, \dots, 6$, let $p^{(i)}(x) = (x-1)(x-i)^2$, $q^{(i)}(x) = (6-x)(x-i)^2$, and define two additional polynomials $p^{(0)}(x) = (x-1)$ and $q^{(0)}(x) = (6-x)$. Now let S be the set consisting of the following orthogonal pairs:

- $\big(p^{(i)}, \mathbf{c}_{2n+2}(i)\big) \quad i = 1, \dots, 6,$
- $\big(q^{(i)}, \mathbf{c}_{2n+2}(i)\big) \quad i = 1, \dots, 6,$
- $\big(p^{(0)}, \mathbf{c}_{2n+2}(1)\big),$
- $\big(q^{(0)}, \mathbf{c}_{2n+2}(6)\big).$

The fact that the matrix T defined in the template using the above pairs indeed has rank 14 can be verified by direct calculation. ■

Theorem 24 *The cone $\mathcal{P}_{2n+2}^{[a,b]}$ is not algebraic. More specifically, for every n, $\beta(\mathcal{P}_{2n+2}^{[a,b]}) \leq 2$.*

Proof: For $n = 1$, using Proposition 21, our claim follows from the previous lemma. From now on, let us assume $n \geq 2$. Consider the matrix valued functions $M\colon \mathbb{R}^{2n+2} \mapsto \mathbb{R}^{(2n+2)\times(2n+2)}$, where the entries of $M(t_1, \dots, t_n; s_1, \dots, s_n; \alpha, \beta) = (m_{i,j})_{i,j=0}^{2n+1}$ are defined as

$$M(t_1, \dots, t_n; s_1, \dots, s_n; \alpha, \beta) = \mathbf{c}_{2n+2}(t_1)\mathbf{p}^\top + \mathbf{c}_{2n+2}(s_1)\mathbf{r}^\top + \alpha\mathbf{c}_{2n+2}(a)\mathbf{p}_a^\top + \beta\mathbf{c}_{2n+2}(b)\mathbf{p}_b^\top,$$

where $\mathbf{p}, \mathbf{r}, \mathbf{p}_a, \mathbf{p}_b \in \mathcal{P}_{2n+2}^{[a,b]}$ are the coefficient vectors of the polynomials $p(x) = (x-a)\prod_{k=1}^{n}(x - t_k)^2$, $r(x) = (b-x)\prod_{k=1}^{n}(x-s_k)^2$, $p_a(x) = x - a$, and $p_b(x) = b - x$, respectively. The entries of $M = (m_{i,j})_{i,j=0}^{2n+1}$ satisfy the polynomial equation

$$\sum_{j=0}^{2n+1} m_{i,j}x^j \equiv t_1^i(x-a)\prod_{k=1}^{n}(x-t_k)^2 + s_1^i(b-x)\prod_{k=1}^{n}(x-s_k)^2 + \alpha a^i(x-a) + \beta b^i(b-x). \quad (11)$$

The polynomials $p(x)$, $p_a(x)$, $p_b(x)$, and $r(x)$ are nonnegative over $[a,b]$, and by Proposition 15,

$$\langle \mathbf{p}_a, \mathbf{c}_{2n+2}(a)\rangle = \langle \mathbf{p}_b, \mathbf{c}_{2n+2}(b)\rangle = \langle \mathbf{p}, \mathbf{c}_{2n+2}(t_1)\rangle = \langle \mathbf{r}, \mathbf{c}_{2n+2}(s_1)\rangle = 0.$$

Consequently, according to the general template of Section 2, the theorem follows if $\dim(\mathrm{span}(M(\mathbb{R}^{2n+2}))) = (2n+2)^2 - 2$. Finally, we will show this equality using the sufficient condition presented in Corollary 19.

Let us define the $(2n+2)$-variate polynomials $\Pi_1(k,\ell)$ and $\Pi_2(k,\ell)$ by

$$\begin{aligned} &\Pi_1(k,\ell)(t_1,\dots,t_n;s_1,\dots,s_n;\alpha,\beta) \overset{\text{def}}{=} \\ &\qquad \Pi(k,\ell)(t_1,\dots,t_n) - a\,\Pi(k,\ell-1)(t_1,\dots,t_n), \text{ and} \\ &\Pi_2(k,\ell)(t_1,\dots,t_n;s_1,\dots,s_n;\alpha,\beta) \overset{\text{def}}{=} \\ &\qquad b\,\Pi(k,\ell-1)(s_1,\dots,s_n) - \Pi(k,\ell)(s_1,\dots,s_n), \end{aligned} \quad (12)$$

where $\Pi(k,\ell)$ is defined by Equation (4) for $0 \le k \le 2n+3$, $0 \le \ell \le 2n-2$, otherwise $\Pi(k,\ell) = 0$.

For the rest of the proof, let us assume that $a = 0$, $b = 1$; Proposition 21 guarantees that this is without loss of generality.

Let $\mathcal{M}$ be the set of entries of M, and let $\mathcal{B}$ denote the set

$$\begin{aligned} \mathcal{B} = \big\{\alpha,\beta\big\} &\cup \big\{\Pi(k,\ell)(t_1,\dots,t_n) \mid 3 \le k \le 2n+3, 0 \le \ell \le 2n-2, k+\ell \ge 2n+1\big\} \cup \\ &\cup \big\{\Pi(k,\ell)(s_1,\dots,s_n) \mid 3 \le k \le 2n+3, \ell = 2n-2\big\} \cup \\ &\cup \big\{\Pi(k,\ell)(s_1,\dots,s_n) \mid 2n \le k \le 2n+1, 0 \le \ell \le 1\big\} \cup \\ &\cup \big\{\Pi(k,\ell)(s_1,\dots,s_n) \mid 0 \le k \le 2n-1, 0 \le \ell \le 2n-2, k+\ell \le 2n\big\}. \end{aligned}$$

Since $n \ge 2$, the last three sets in the union are disjoint. It follows from the definition that the set $\mathcal{B}$ is linearly independent, because no two polynomials share a common monomial. The coefficient of $x^{2n+1-k-\ell}$ in the polynomial $(x-a)\prod_{j=1}^{n}(x-t_j)^2$ is $\sum_{k=0}^{2}\Pi_1(k,\ell)$. Similarly, the coefficient of $x^{2n+1-k-\ell}$ in the polynomial $(b-x)\prod_{j=1}^{n}(x-s_j)^2$ is $\sum_{k=0}^{2}\Pi_2(k,\ell)$. From this observation it follows immediately that $\mathrm{span}(\mathcal{B})$ contains the entries of M; more specifically, for every $0 \le i,j \le 2n+1$,

$$\begin{aligned} m_{i,j} = {} & \Pi_1(i,2n+1-j) + \Pi_1(i+1,2n-j) + \Pi_1(i+2,2n-1-j) + \\ & + \Pi_2(i,2n+1-j) + \Pi_2(i+1,2n-j) + \Pi_2(i+2,2n-1-j) + \\ & + m'_{i,j}, \end{aligned} \quad (13)$$

where $m'_{i,j}$ is defined in Equation (9).

We now introduce an injection $\varphi\colon \mathcal{B} \mapsto \mathcal{M}$ by defining its inverse (where it exists): let $m_{i,j}$ be the image of the polynomial

$$\varphi^{-1}(m_{i,j}) = q_{i,j} = \begin{cases} \alpha & i = 0, j = 1 \\ \Pi(i+2, 2n-1-j)(t_1, \ldots, t_n) & 1 \le j \le \min\{i, 2n-1\} \\ \Pi(i+2, 2n-2)(s_1, \ldots, s_n) & i \ge 1, j = 0 \\ \beta & i = 0, j = 0 \\ \Pi(i, 2n+1-j)(s_1, \ldots, s_n) & j \ge 3, j > \min\{i, 2n-1\} \\ \text{not defined} & \text{otherwise} \end{cases} . \quad (14)$$

In particular, we assign a polynomial to each entry except for $m_{0,2}$ and $m_{1,2}$, and we assign different polynomials to different entries of M, by an argument essentially identical to that in the proof of Theorem 20. Consequently, each polynomial in $\mathcal{B}$ is equal to $q_{i,j}$ for at most one pair (i, j), and Equations (13) and (14) show that (assuming $a = 0$ and $b = 1$) the coefficient of $q_{i,j}$ in the $\mathcal{B}$-representation of $m_{i,j}$ is 1 or -1.

Let us define a linear order $\succ$ on $\varphi(\mathcal{B})$ in the following way. Let us say that a polynomial $q_{i,j}$ is *of type* k for some $k = 1, \ldots, 5$, if it is defined in the kth branch of the right-hand side of (14). Then, $m_{i_1,j_1} \succ m_{i_2,j_2}$ for some $(i_1, j_1) \neq (i_2, j_2)$ precisely when one of the following four conditions holds:

1. q_{i_1,j_1} is of smaller type than q_{i_2,j_2};

2. q_{i_1,j_1} and q_{i_2,j_2} are both of type 2, and either $i_1 > i_2$, or $i_1 = i_2$ but $j_1 < j_2$;

3. q_{i_1,j_1} and q_{i_2,j_2} are both of type 3, and $i_1 > i_2$;

4. q_{i_1,j_1} and q_{i_2,j_2} are both of type 5, and either $j_1 < j_2$, or $j_1 = j_2$ but $i_1 > i_2$.

Clearly this is indeed a linear order on $\varphi(\mathcal{B})$. An easy case analysis using (13), (9), and (14) shows that if $m_{i_1,j_1} \succ m_{i_2,j_2}$, then the coefficient of q_{i_1,j_1} in the $\mathcal{B}$-representation of m_{i_2,j_2} is zero:

1. If $(i_1, j_1) = (0, 1)$, then $q_{i_1,j_1} = \alpha$, and this polynomial has a nonzero coefficient exclusively in the $\mathcal{B}$-representation of $m_{0,1}$. In the remaining cases we assume $(i_1, j_1) \neq (0, 1)$.

2. The case $(i_2, j_2) = (0, 1)$ is impossible.

3. If $(i_1, j_1) = (0, 0)$, then $m_{i_1,j_1} \succ m_{i_2,j_2}$ implies $j_2 \ge 3$ (with a similar argument as in the proof of Theorem 22), so $q_{0,0} = \beta$ has zero coefficient in the $\mathcal{B}$-representation of m_{i_2,j_2}.

4. If $(i_2, j_2) = (0, 0)$, then $m_{i_1,j_1} \succ m_{i_2,j_2}$ implies $i_1 \ge 1$, so $q_{i_1,j_1} = \Pi(k, \ell)$ with some $k \ge 3$. Consequently, q_{i_1,j_1} has zero coefficient in the $\mathcal{B}$-representation of $m_{0,0}$. In the remaining cases we assume both (i_1, j_1) and (i_2, j_2) are different from $(0, 0)$ and $(0, 1)$.

5. The cases in which q_{i_1,j_1} and q_{i_2,j_2} are of the same type are settled the same way as in the last two cases of the case analysis in the proof of Theorem 20.

6. The case when q_{i_1,j_1} is of type 2 or 3, and q_{i_2,j_2} is of type 4, is settled the same way as in the proof of Theorem 20, by a simple degree argument.

7. The only remaining case is when q_{i_1,j_1} is of type 2 and q_{i_2,j_2} is of type 3. Then $i_1+2 > i_2+2$, and hence $q_{i_1,j_1} = \Pi(i_1+2, 2n-1-j)$ has coefficient zero in the $\mathcal{B}$-representation of m_{i_2,j_2}.

We conclude that the injection $m_{i,j} \mapsto q_{i,j}$ and the linear order $\succ$ satisfy the conditions of Corollary 19, therefore, by Equation (14),

$$\dim(\operatorname{span}(M(\mathbb{R}^{2n+2}))) = |\mathcal{B}| = (2n+2)^2 - 2,$$

which completes the proof. ■

4.3 Lower bounds

To simplify the proof of the validity of bilinear optimality conditions, we will use the following lemma.

Lemma 25 *The bilinear optimality condition $\mathbf{x}^\top Q\mathbf{s} = 0$ is satisfied by every $(\mathbf{x}, \mathbf{s}) \in C(\mathcal{K})$ if and only if it is satisfied by every $(\mathbf{x}, \mathbf{s}) \in C(\mathcal{K})$ such that $\mathbf{x}$ is an extreme vector of $\mathcal{K}$ and $\mathbf{s}$ is an extreme vector of $\mathcal{K}^*$.*

Proof: The *only if* direction is obvious. To show the converse implication, observe that every $\mathbf{x} \in \mathcal{K}$ and $\mathbf{s} \in \mathcal{K}^*$ can be expressed as a sum of finitely many extreme vectors of $\mathcal{K}$ and $\mathcal{K}^*$, respectively. Furthermore, if $\mathbf{x} = \sum_{i=1}^k \mathbf{x}_i$ and $\mathbf{s} = \sum_{j=1}^\ell \mathbf{s}_j$, then $\langle \mathbf{x}, \mathbf{s} \rangle = 0$ if and only if $\langle \mathbf{x}_i, \mathbf{s}_j \rangle = 0$ for every $1 \le i \le k$, $1 \le j \le \ell$. Therefore, if $\langle \mathbf{x}, \mathbf{s} \rangle = 0$, and the optimality condition is satisfied by every orthogonal pair of extreme vectors, then $\langle \mathbf{x}_i, \mathbf{s}_j \rangle = 0$ for every $1 \le i \le k$, $1 \le j \le \ell$, and

$$\mathbf{x}^\top Q\mathbf{s} = \left(\sum_{i=1}^k \mathbf{x}_i\right)^\top Q \left(\sum_{j=1}^\ell \mathbf{s}_j\right) = \sum_{i=1}^k \sum_{j=1}^\ell \mathbf{x}_i^\top Q\mathbf{s}_j = 0.$$

■

We are now ready to show that the upper bounds on the number of linearly independent bilinear optimality conditions given in Theorems 20, 22, and 24 are sharp.

Theorem 26 *For every integer $n \ge 1$, $\beta(\mathcal{P}_{2n+1}) = 4$.*

Proof: We have already proven $\beta(\mathcal{P}_{2n+1}) \leq 4$. Now we prove that the following bilinear optimality conditions satisfied by every $(\mathbf{p}, \mathbf{c}) \in C(\mathcal{P}_{2n+1})$:

$$\sum_{i=0}^{2n} p_i c_i = 0, \tag{15}$$

$$\sum_{i=1}^{2n} i p_i c_{i-1} = 0, \tag{16}$$

$$\sum_{i=0}^{2n-1} (2n-i) p_i c_i = 0, \tag{17}$$

$$\sum_{i=0}^{2n-1} (2n-i) p_i c_{i+1} = 0. \tag{18}$$

It is easy to see that these conditions are indeed linearly independent. By Lemma 25 it is enough to show that the conditions are satisfied for pairs of vectors $(\mathbf{p}, \mathbf{c}) \in C(\mathcal{P}_{2n+1})$ where $\mathbf{c}$ is an extreme vector of $\bar{\mathcal{M}}_{2n+1}$.

If $\mathbf{c} = c_{2n}\mathbf{e}_{2n} = (0, \ldots, 0, c_{2n})$ with some $c_{2n} > 0$ and $\langle \mathbf{p}, \mathbf{e}_{2n} \rangle = 0$, then (15), (16), and (17) trivially hold, since all the terms on the left-hand sides these equations are zeros. Furthermore, the left-hand side of (18) simplifies to $p_{2n-1}c_{2n}$, which must be zero, because otherwise $p_{2n-1} \neq 0$, $p_{2n} = 0$, and p would be a polynomial of odd degree, which cannot be nonnegative over the entire real line.

If $\mathbf{c}$ is an extreme vector of $\mathcal{M}_{2n+1}$, then, by Proposition 15, $\mathbf{c} = \mathbf{c}(t_0)$ for some $t_0 \in \mathbb{R}$, and $\mathbf{c}$ is orthogonal to $\mathbf{p}$ if and only if $p(t_0) = 0$. But this equation is equivalent to (15), since

$$p(t_0) = \sum_{i=0}^{2n} p_i t_0^i = \sum_{i=0}^{2n} p_i c_i.$$

By Proposition 17, every root of p has even multiplicity, therefore $(\mathbf{p}, \mathbf{c}) \in C(\mathcal{K})$ implies $p'(t_0) = 0$, which is equivalent to (16), as

$$p'(t_0) = \sum_{i=1}^{2n} p_i i t_0^{i-1} = \sum_{i=1}^{2n} i p_i c_{i-1}.$$

Furthermore, if $p(t_0) = p'(t_0) = 0$, then $2np(t_0) - t_0 p'(t_0) = 0$, which translates to (17), since

$$2np(t_0) - t_0 p'(t_0) = \sum_{i=0}^{2n} 2n p_i t_0^i - \sum_{i=1}^{2n} p_i i t_0^i = \sum_{i=0}^{2n} 2n p_i c_i - \sum_{i=1}^{2n} i p_i c_i = \sum_{i=0}^{2n} (2n-i) p_i c_i.$$

Finally, $p(t_0) = p'(t_0) = 0$ also implies $2nt_0 p(t_0) - t_0^2 p'(t_0) = 0$, which is equivalent to (18):

$$2nt_0 p(t_0) - t_0^2 p'(t_0) = \sum_{i=0}^{2n} 2n p_i t_0^{i+1} - \sum_{i=1}^{2n} p_i i t_0^{i+1} = \sum_{i=0}^{2n} 2n p_i c_{i+1} - \sum_{i=1}^{2n} i p_i c_{i+1} = \sum_{i=0}^{2n-1} (2n-i) p_i c_{i+1}.$$

■

Theorem 27 *For every integer $n \geq 1$ and real numbers $a < b$, $\beta(\mathcal{P}_{n+1}^{[a,b]}) = 2$.*

Proof: We have already proven $\beta(\mathcal{P}_{n+1}^{[a,b]}) \leq 2$. Now we prove that the following bilinear optimality conditions satisfied by every $(\mathbf{p}, \mathbf{c}) \in C(\mathcal{P}_{n+1}^{[a,b]})$:

$$\sum_{i=0}^{n} p_i c_i = 0, \tag{19a}$$

$$\sum_{i=0}^{n-1} \big((a+b)(n-i)p_i c_i - (n-i)p_i c_{i+1} + ab(i+1)p_{i+1}c_i\big) = 0. \tag{19b}$$

It is easy to see that these conditions are indeed linearly independent. By Lemma 25 it is enough to show that the conditions are satisfied for pairs of vectors $(\mathbf{p}, \mathbf{c}) \in C(\mathcal{P}_{n+1}^{[a,b]})$ where $\mathbf{c}$ is an extreme vector of $\bar{\mathcal{M}}_{n+1}$.

Let $p(x)$ be an extreme polynomial of degree n, nonnegative over $[a, b]$, and $\mathbf{p}$ its coefficient vector. By Proposition 15, an extreme $\mathbf{c} = \mathbf{c}(t_0)$ is orthogonal to $\mathbf{p}$ if and only if $p(t_0) = 0$. Therefore, $(\mathbf{p}, \mathbf{c}) \in C(\mathcal{P}_{n+1}^{[a,b]})$ implies $p(t_0) = 0$, which is equivalent to (19a), as in the proof of the previous theorem. By Proposition 17, every root of p has even multiplicity, except possibly for a and b, and hence

$$(t_0 - a)(b - t_0)p'(t_0) = 0.$$

Finally, this last equality and $p(t_0) = 0$ together imply

$$n(a + b - t_0)p(t_0) - (t_0 - a)(b - t_0)p'(t_0) = 0,$$

equivalent to (19b):

$$\begin{aligned}
n(a+b-t_0)p(t_0)-(t_0-a)(b-t_0)p'(t_0) &= \\
&= \sum_{i=0}^{n} n(a+b-t_0)p_i t_0^i - \sum_{i=1}^{n}(t_0-a)(b-t_0)ip_i t_0^{i-1} \\
&= \sum_{i=0}^{n} n(a+b)p_i t_0^i - \sum_{i=0}^{n} np_i t_0^{i+1} + \sum_{i=1}^{n} ip_i t_0^{i+1} - \sum_{i=1}^{n}(a+b)ip_i t_0^i + \sum_{i=1}^{n} abip_i t_0^{i-1} \\
&= \sum_{i=0}^{n-1}(n-i)(a+b)p_i t_0^i - \sum_{i=0}^{n-1}(n-i)p_i t_0^{i+1} + \sum_{i=1}^{n} abip_i t_0^{i-1} \\
&= \sum_{i=0}^{n-1} \big((n-i)(a+b)p_i c_i - (n-i)p_i c_{i+1} + ab(i+1)p_{i+1}c_i\big).
\end{aligned}$$

■

5 More non-algebraic cones

As we have seen in Lemma 9, applying a nonsingular linear transformation A to a cone $\mathcal{K}$ preserves its bilinearity rank. In addition, any change of basis for the cone of polynomials will result in a cone algebraically equivalent to it. Thus, for instance, the set of vectors of coefficients of nonnegative polynomials expressed in any orthogonal polynomial basis (e.g., Laguerre, Legendre, Chebyshev, etc.), or the Bernstein polynomial basis $\binom{n}{k}t^k(1-t)^{n-k}$ for $k = 0, \ldots, n$ are algebraically equivalent to the cone of nonnegative polynomials in the standard basis. This fact is useful in numerical computations since the standard basis is numerically unstable and we may need to work with a more stable basis.

We have already stated in Proposition 21 that for all $a < b$ and n, the cones $\mathcal{P}^{[a,b]}_{n+1}$ and $\mathcal{P}^{[0,1]}_{n+1}$ are algebraically equivalent. More generally:

Observation 28 *Let $f(t)$ be a function whose domain is $\Delta \subseteq \mathbb{R}$ and whose range is $\Omega \subseteq \mathbb{R}$; also suppose that the set of functions $\{1, f, f^2, \ldots, f^n\}$ is linearly independent. Then the cone*

$$\mathcal{P}^f = \left\{ \mathbf{a} = (a_0, \ldots, a_n) \;\middle|\; \sum_{j=0}^{n} a_j f^j(t) \geq 0 \ \ \textit{for all } t \in \Delta \right\}$$

is algebraically equivalent to the cone of ordinary polynomials nonnegative over Ω.

From this observation and using change of basis as needed we can prove algebraic equivalence of a number of cones of nonnegative functions over well-known finite dimensional bases with $\mathcal{P}$ or $\mathcal{P}^{[0,1]}$. Below we present a partial list. Most of the techniques used below are quite simple, and they are used by Karlin and Studden [7] and Nesterov [9] for other purposes.

5.1 Rational functions

The basis $\{t^{-m}, t^{-m+1}, \ldots, t^{n-1}, t^n\}$ for nonnegative even integers n and m spans the set of rational functions with a degree n numerator and denominator t^m. Since $\sum_{i=-m}^{n} p_i t^i = t^{-m} \sum_{i=0}^{n+m} p_i t^i$, the cone of rational functions with numerator of degree n and denominator t^m nonnegative over $\Delta = \mathbb{R}\backslash\{0\}$ is algebraically equivalent to the cone of nonnegative polynomials of degree $n + m$, and therefore its bilinearity rank is 4.

5.2 Nonnegative polynomials over $[0, \infty)$

Consider the basis $B = \{t^n, t^{n-1}(1-t), \ldots, t(1-t)^{n-1}, (1-t)^n\}$ of polynomials of degree n. Clearly the cone

$$\left\{ (p_0, \ldots, p_n) \;\middle|\; \sum_{i=0}^{n} p_i t^i (1-t)^{n-i} \geq 0 \quad \forall\, t \in [0,1] \right\},$$

which consists of coefficient vectors of polynomials nonnegative over $[0, 1]$, expressed in basis B, is algebraically equivalent to $\mathcal{P}^{[0,1]}_{n+1}$. On the other hand we have:

Lemma 29 (Nesterov [9]) *A polynomial $p_0(1-t)^n + p_1 t(1-t)^{n-1} + \cdots + p_n t^n$ is nonnegative over $[0, 1]$ if and only if the polynomial $p_0 + p_1 t + \cdots + p_n t^n$ is nonnegative over $[0, \infty]$.*

This follows from

$$\sum_k p_k t^k (1-t)^{n-k} = (1-t)^n \sum_k p_k \left(\tfrac{t}{1-t}\right)^k$$

and the fact that $[0,1]$ is mapped to $[0,\infty]$ under $f(t) = \frac{t}{1-t}$. As a result we get

Corollary 30 *The cone $\mathcal{P}_{n+1}^{[0,\infty]}$ is algebraically equivalent to $\mathcal{P}_{n+1}^{[0,1]}$. Therefore $\beta\big(\mathcal{P}^{[0.\infty]}\big) = 2$ for every $n \in \mathbb{N}$.*

5.3 Cosine polynomials

Consider the cone

$$\mathcal{P}_{n+1}^{\cos} \stackrel{\text{def}}{=} \{\mathbf{c} \in \mathbb{R}^{n+1} \mid \sum_{k=0}^{n} c_k \cos(kt) \geq 0 \text{ for all } t \in \mathbb{R}\}$$

To relate this cone to the cones we have discussed before, first observe that $\cos(kt)$ can be expressed as an ordinary polynomial of degree k of $\cos(t)$. This follows immediately from applying the binomial theorem to the identity $\big(\cos(t) + i\sin(t)\big)^k = \cos(kt) + i\sin(kt)$:

$$\cos(kt) = \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j} \cos^{k-2j}(t)\big(1-\cos^2(t)\big)^j \tag{20}$$

$$\sin(kt) = \sin(t) \sum_{j=0}^{\lfloor k/2 \rfloor} \binom{k}{2j+1} \cos^{k-2j-1}(t)\big(1-\cos^2(t)\big)^j \tag{21}$$

From (20) we see that expansion of $\cos(kt)$ is a polynomial of $\cos(t)$ of degree k. Thus, every vector $\mathbf{c}$ representing the cosine polynomial $c(t) = \sum_{k=0}^{n} c_k \cos(kt)$ is mapped to a vector $\mathbf{p}$ representing the ordinary polynomial $p(s) = \sum_{k=0}^{n} p_k s^k$ through the identity $\sum_{k=0}^{n} c_k \cos(kt) = p\big(\cos(t)\big)$. Furthermore, this correspondence between $\mathbf{c}$ and $\mathbf{p}$ is one-to-one and onto, since for each k the function $\cos(kt)$ is a polynomial of degree k in $\cos(t)$ the matrix mapping $\mathbf{c}$ to $\mathbf{p}$ is lower triangular with nonzero diagonal entries. Now $c(t) = p\big(\cos(t)\big) \geq 0$ for all t if and only if $p(s) \geq 0$ for all $s \in [-1,1]$. Recalling that $\mathcal{P}^{[-1,1]}$ is algebraically equivalent to $\mathcal{P}^{[0,1]}$ we have proved

Lemma 31 *The cone $\mathcal{P}_{n+1}^{\cos}$ is algebraically equivalent to $\mathcal{P}_{n+1}^{[0,1]}$. Therefore $\beta\big(\mathcal{P}_{n+1}^{\cos}\big) = 2$ for every $n \in \mathbb{N}$.*

5.4 Trigonometric polynomials

Consider the cone

$$\begin{aligned}
\mathcal{P}_{2n+1}^{\text{trig}} &= \Big\{\mathbf{r} \in \mathbb{R}^{2n+1} \mid r_0 + \sum_{k=1}^{n} \big(r_{2k-1}\cos(kt) + r_{2k}\sin(kt)\big) \geq 0 \text{ for all } t \in \mathbb{R}\Big\} \\
&= \Big\{\mathbf{r} \in \mathbb{R}^{2n+1} \mid r_0 + \sum_{k=1}^{n} \big(r_{2k-1}\cos(kt) + r_{2k}\sin(kt)\big) \geq 0 \text{ for all } t \in (-\pi,\pi)\Big\}
\end{aligned}$$

To transform a trigonometric polynomial $r(t) = r_0 + \sum_{k=1}^{n} \big(r_{2k-1}\cos(kt) + r_{2k}\sin(kt)\big)$ into one of the classes of polynomials already discussed we make a change of variables $t = 2\arctan(s)$. With this transformation we have

$$\sin(t) = \frac{2s}{1+s^2}$$
$$\cos(t) = \frac{1-s^2}{1+s^2}$$

Using (20-21) we can write

$$r(t) = p_1\big(\tfrac{1-s^2}{1+s^2}\big) + \tfrac{2s}{1+s^2} p_2\big(\tfrac{1-s^2}{1+s^2}\big) \tag{22}$$

where p_1 and p_2 are ordinary polynomials of degree n, and $n-1$, respectively; $p_1(\cdot)$ is obtained from (20) and $p_2(\cdot)$ is obtained from (21). Multiplying by $(1+s^2)^n$ we see that

$$r(t) = (1+s^2)^{-n} p(s)$$

for some ordinary polynomial p. Substituting (20) and (21), the polynomial p can be expressed in the following basis:

$$\Big\{(1+s^2)^n, (1+s^2)^{n-1}(1-s^2), \ldots, (1-s^2)^n\Big\} \cup \Big\{s(1+s^2)^{n-1}, s(1-s^2)^{n-2}(1-s^2), \ldots, s(1-s^2)^{n-1}\Big\}$$

It is straightforward to see that this is indeed a basis. We need to simply observe that those terms that are not multiplied by s form a basis of polynomials with even degree terms, and those that involve s form a basis of polynomials with odd degree terms. Therefore, the correspondence between vector of coefficients $\mathbf{r}$ of the trigonometric polynomial $r(t)$ and the vector of coefficients $\mathbf{p}$ of the ordinary polynomial $p(s)$ in the above basis is one-to-one and onto. Furthermore, since the function $\tan(t/2)$ maps $(-\pi, \pi)$ to $\mathbb{R}$, it follows that

Lemma 32 *The cone $\mathcal{P}_{n+1}^{trig}$ is algebraically equivalent to $\mathcal{P}_{n+1}$. Therefore, $\beta\big(\mathcal{P}_{n+1}^{trig}\big) = 4$ for every $n \in \mathbb{N}$.*

5.5 Exponential polynomials

One could ask if the results of trigonometric polynomials extend to hyperbolic functions $\sinh(\cdot)$ and $\cosh(\cdot)$. The situation is actually simpler here. Consider the cone

$$\mathcal{P}^{\exp} \stackrel{\text{def}}{=} \Big\{\mathbf{e} \mid \sum_{k=-m}^{n} e_k \exp(kt) \geq 0 \text{ for all } t \geq 0\Big\}$$

First, there is no loss of generality if we assume $m = 0$ since every such polynomial can be mutltiplied by $\exp(mt)$. Now clearly $e(t) = e_0 + e_1\exp(t) + \cdots + e_n\exp(nt) \geq 0$ for all $t \in \mathbb{R}$ if and only if the ordinary polynomial $e_0 + e_1 s + \cdots + e_n s^n$ is nonnegative over $[0, \infty]$. Recalling that $\mathcal{P}^{[0,\infty]}$ is algebraically equivalent to $\mathcal{P}^{[0,1]}$ we have shown that

Lemma 33 *The cone $\mathcal{P}^{\exp}$ is algebraically equivalent to $\mathcal{P}^{[0,1]}$. Therefore, $\beta\big(\mathcal{P}^{\exp}\big) = 2$.*

6 Conclusion

Our main motivation for this research came from our work on solving statistical nonparametric estimation problems using polynomials and polynomial splines where the estimated functions themselves required to be nonnegative, [2] and [11]. Our goal was to see if there is an easier way than formulating these problems as semidefinite programs. In particular are there efficient algorithms for cone-LP problems over positive polynomials? This questions led us to consider the simplest form of complementarity relations for positive polynomials, and we have found that bilinear complementarity relations alone are not sufficient.

The central question remaining open is whether there are algebraic cones other than symmetric cones and their algebraic equivalents?

Another direction is to investigate more sets of cones and estimate their bilinearity rank. For example one can examine all cones of positive functions over Chebyshev systems, and cones of functions of several variables which can be expressed as sums of squares of functions over a given finite set of functions.

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Appendix: Proof of Theorem 2, due to O. Güler [6]

Recall the following basic fact.

Proposition 34 *Let $S \subseteq R^n$ be a closed convex set and $\mathbf{a} \in \mathbb{R}^n$. Then there is a unique point $\mathbf{x} = \Pi_S(\mathbf{a})$ in S which is closest to $\mathbf{a}$, i.e., there is a unique point $\mathbf{x} \in S$ such that $\mathbf{x} = \operatorname{argmin}_{\mathbf{y}\in S} \|\mathbf{a} - \mathbf{y}\|$. Furthermore, if S is a closed convex cone, then $\langle \mathbf{x}, \mathbf{x} - \mathbf{a} \rangle = 0$.*

The unique point above is called the *projection of* $\mathbf{a}$ *to* S.

We need to show a continuous bijection between the complementarity set $C(\mathcal{K})$ of $\mathcal{K}$ and $\mathbb{R}^n$ whose inverse is also continuous.

Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ be defined by $\varphi(\mathbf{a}) = (\mathbf{x}, \mathbf{s})$, where $\mathbf{x} = \Pi_{\mathcal{K}}(\mathbf{a})$ and $\mathbf{s} = \mathbf{x} - \mathbf{a}$. Clearly φ is continuous; we first show that $\varphi(\mathbf{a}) \in C(\mathcal{K})$ for every $\mathbf{a}$. By definition $\Pi_{\mathcal{K}}(\mathbf{a}) \in \mathcal{K}$, and by the above proposition $\langle \mathbf{x}, \mathbf{s} \rangle = 0$. It remains to show that $\mathbf{s} \in \mathcal{K}^*$.

For an arbitrary $\mathbf{u} \in \mathcal{K} \setminus \{\mathbf{x}\}$, define the convex combination $\mathbf{u}_\alpha = \alpha\mathbf{u} + (1-\alpha)\mathbf{x}$ where $0 \le \alpha \le 1$, and let $\zeta(\alpha) = \|\mathbf{a} - \mathbf{u}_\alpha\|^2$. Then ζ is a differentiable function on the interval $[0, 1]$, and $\min_{0\le\alpha\le1} \zeta(\alpha)$ is attained at $\alpha = 0$. Hence $\frac{d\zeta}{d\alpha}\Big|_{\alpha=0} \ge 0$.

Now, using $\langle \mathbf{x}, \mathbf{s} \rangle = 0$, we have

$$\frac{d\zeta}{d\alpha}\bigg|_{\alpha=0} = 2\langle \mathbf{s}, \mathbf{u} - \mathbf{x} \rangle = 2\langle \mathbf{s}, \mathbf{u} \rangle \ge 0$$

for every $\mathbf{u} \in \mathcal{K} \setminus \{\mathbf{x}\}$. Note that the inequality $\langle \mathbf{s}, \mathbf{u} \rangle \ge 0$ also holds for $\mathbf{u} = \mathbf{x}$, implying $\langle \mathbf{s}, \mathbf{u} \rangle \ge 0$ for every $\mathbf{u} \in \mathcal{K}$. Therefore $\mathbf{s} \in \mathcal{K}^*$.

Consider now the continuous function $\bar{\varphi}\colon C(\mathcal{K}) \to \mathbb{R}^n$ defined by $\bar{\varphi}(\mathbf{x}, \mathbf{s}) = \mathbf{x} - \mathbf{s}$. To conclude the proof we show that $\bar{\varphi} \circ \varphi = \iota_{\mathbb{R}^n}$ and $\varphi \circ \bar{\varphi} = \iota_{C(\mathcal{K})}$, where ι_S denotes the identity function of the set S. The first one is easy:

$$(\bar{\varphi} \circ \varphi)(\mathbf{a}) = \bar{\varphi}\left(\Pi_{\mathcal{K}}(\mathbf{a}), \Pi_{\mathcal{K}}(\mathbf{a}) - \mathbf{a}\right) = \mathbf{a}.$$

To show $\varphi \circ \bar{\varphi} = \iota_{C(\mathcal{K})}$, it suffices to prove that $\Pi_{\mathcal{K}}(\mathbf{x} - \mathbf{s}) = \mathbf{x}$ for every $(\mathbf{x}, \mathbf{s}) \in C(\mathcal{K})$.

Suppose on the contrary that there is a point $\mathbf{u} \in \mathcal{K}$ such that $\|\mathbf{a} - \mathbf{u}\| < \|\mathbf{a} - \mathbf{x}\|$, where $\mathbf{a} = \mathbf{x} - \mathbf{s}$. Then, again using $\langle \mathbf{x}, \mathbf{s} \rangle = 0$,

$$0 > \langle \mathbf{a} - \mathbf{u}, \mathbf{a} - \mathbf{u} \rangle - \langle \mathbf{a} - \mathbf{x}, \mathbf{a} - \mathbf{x} \rangle = \langle \mathbf{x} - \mathbf{s} - \mathbf{u}, \mathbf{x} - \mathbf{s} - \mathbf{u} \rangle - \langle \mathbf{s}, \mathbf{s} \rangle = \|\mathbf{x} - \mathbf{u}\|^2 + 2\langle \mathbf{s}, \mathbf{u} \rangle,$$

in contradiction with $\langle \mathbf{s}, \mathbf{u} \rangle \geq 0$, which completes the proof. ■