OPTIMIZATION & NONLIN EQS (MA 784) — HW 1

Fundamentals

Due on January 29 (Thursday), by the start of the class. Please turn in your solutions to me, on paper. (*Hints are available if needed, but give it a serious try by yourself first.*)

1. Construct a polynomial that has a finite, unattained infimum.

2. Determine the gradient of the function $\ln(\det(\cdot))$ at $\mathbf{X}$, assuming that $\mathbf{X}$ is a matrix with positive determinant (i.e., the function is well-defined).

3. In this sequence of questions, you will derive Farkas' Lemma from first principles (without using any convex analysis or even linear algebra). There are many equivalent forms of this statement; here is one:

Theorem (Farkas' Lemma) *Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Then exactly one of the following statements holds:*

(a) There exists an $\mathbf{x} \in \mathbb{R}^n$ satisfying $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq 0$.

(b) There exists a $\mathbf{y} \in \mathbb{R}^m$ satisfying $\mathbf{A}^{\mathrm{T}}\mathbf{y} \geq 0$ and $\mathbf{b}^{\mathrm{T}}\mathbf{y} < 0$.

The approach is to reduce this to the mutually exclusive possibilities of the minimum of an unconstrained optimization problem being 0 or not. Throughout, let $\|\cdot\|$ denote the Euclidean norm (2-norm), and let $(\cdot)_- : \mathbb{R}^n \to \mathbb{R}^n$ denote the componentwise *negative part* function defined as

$$(\mathbf{x})_- = \begin{bmatrix} -\min(x_1, 0) \\ -\min(x_2, 0) \\ \vdots \\ -\min(x_n, 0) \end{bmatrix}.$$

(a) Consider the $\mathbb{R}^n \to \mathbb{R}$ function f defined by

$$f(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|^2 + \|(\mathbf{x})_-\|^2.$$

Show that this is a convex and differentiable function that takes only nonnegative values. What is $\nabla f(\mathbf{x})$?

(b) Show that f attains a minimum.

(c) Argue that the minimum value of f is 0 *if and only if* the first alternative of the Theorem holds.

(d) Based on the above, the second alternative should be equivalent to the following scenario: f has a minimizer, but its function value is positive. Use this observation to complete the proof of the theorem.

4. Extra credit (Almost Fermat's last theorem.) Let

$$F(a, b, c, n) = (a^n + b^n - c^n)^2 + \sin(a\pi)^2 + \sin(b\pi)^2 + \sin(c\pi)^2 + \sin(n\pi)^2.$$

Show that

$$\inf\{F(a, b, c, n) \mid a \geq 1, b \geq 1, c \geq 1, n \geq 3\} = 0.$$

(If you can show directly that the infimum is not attained, do let me in on the secret!)

5. Extra credit (A challenging derivative.) Let $X \subseteq \mathbb{R}^n$ be a closed, convex, non-empty set, and $f(\mathbf{x}) \overset{\text{def}}{=} \|\mathbf{x} - \Pi_X(\mathbf{x})\|^2$ the "squared-distance-from-X" function. (Π_X is the orthogonal projection onto X operator.) Show that f is a differentiable function whose gradient is given by $\nabla f(\mathbf{x}) = 2(\mathbf{x} - \Pi_X(\mathbf{x}))$.